What is a Probabilistically Defined Quantum Field Theory?

Sourav Chatterjee

Lagrangian mechanics

Lagrangian mechanics studies the trajectory of a physical system and assigns to it an action.

For example, if $\gamma : \mathbb{R} \to \mathbb{R}^3$ is the trajectory of a particle of mass m moving in a potential V, then the action is formally given by

$$S(\gamma) = \int_{-\infty}^{\infty} \left(\frac{1}{2} m \|\dot{\gamma}(t)\|^2 - V(\gamma(t)) \right) dt.$$

The classical trajectory is required to be a critical point of the action. That is, we need

$$\frac{\delta S}{\delta \gamma} = 0.$$

For this system, the equation becomes $m\ddot{\gamma}(t) = -\nabla V(\gamma(t))$, which is the familiar equation of motion in Newtonian mechanics.

Path integral formulation of quantum mechanics

The passage from classical to quantum mechanics replaces a single classical trajectory by a formal superposition of trajectories.

Rather than trying to compute the value of a real-valued function $F(\gamma)$ on the classical trajectory, we now try to compute its quantum expectation.

In the path integral formulation of quantum mechanics, this is formally expressed as

$$\langle F(\gamma)\rangle = \frac{\int F(\gamma) e^{\frac{\mathrm{i}}{\hbar} S(\gamma)} \mathcal{D}\gamma}{\int e^{\frac{\mathrm{i}}{\hbar} S(\gamma)} \mathcal{D}\gamma},$$

where $\mathrm{i} = \sqrt{-1}$ and $\hbar$ is Planck's constant, and $\int \ldots \mathcal{D}\gamma$ denotes integration over the space of all trajectories, which is often not mathematically well defined.

From quantum to classical

In the classical regime, $\hbar$ is extremely small.

Heuristically, this means that $e^{\frac{\mathrm{i}}{\hbar}S(\gamma)}$ exhibits very large fluctuations if γ is moved slightly away from a critical point of S.

Thus, again heuristically, the main contribution to the path integrals should come from small neighborhoods of the critical points.

So, if there is a unique critical point $\widehat{\gamma}$, then $\langle F(\gamma)\rangle$ should be $\approx F(\widehat{\gamma})$ when $\hbar$ is very small.

State space

Let $\gamma : \mathbb{R} \to \Omega$ be the trajectory of a physical system, where Ω is the space of possible instantaneous configurations of the system.

Let $S(\gamma)$ be the action of the trajectory γ.

Heuristically, if we are interested in evaluating $\langle F(\gamma(t))\rangle$ — that is, the expected value of a function of the trajectory at a fixed instant of time via path integrals — then in stationary situations the answer does not depend on t, and is given by $\int_\Omega F d\mu$ for some probability measure μ on Ω.

In this case, we take $\mathcal{H} = L^2(\mu)$ to be the Hilbert space of the quantum system. Thus elements of $\mathcal{H}$ represent the possible states of the system at an instant of time.

Time evolution groups

Suppose we have such a state space $\mathcal{H} = L^2(\mu)$. Then, again heuristically, one often finds that for any $F, G \in \mathcal{H}$ and $t_1, t_2 \in \mathbb{R}$, the expectation value

$$\langle \overline{F(\gamma(t_1))} G(\gamma(t_2)) \rangle$$

depends only on the difference $t = t_2 - t_1$, and moreover, there exists a unitary operator $U(t) : \mathcal{H} \to \mathcal{H}$ such that

$$\langle \overline{F(\gamma(t_1))} G(\gamma(t_2)) \rangle = \langle U(t) G, F \rangle_{\mathcal{H}}.$$

It is also usually the case that $(U(t))_{t \in \mathbb{R}}$ is a group of unitary operators, meaning that $U(s+t) = U(s)U(t)$ for all $s, t \in \mathbb{R}$.

Constructive field theory

Conversely, if one can construct the group $(U(t))_{t\in\mathbb{R}}$, then one can give meaning to the expectation values defined earlier using path integrals.

Constructive field theory is a branch of mathematical physics, developed in the 1960s and 1970s by many contributors (Wightman, Symanzik, Nelson, Glimm, Jaffe, Osterwalder, Schrader, ...), that seeks to construct $\mathcal{H}$ and $(U(t))_{t\in\mathbb{R}}$ rigorously for various quantum field theories.

One widely used axiomatic framework is given by the Wightman axioms. Another is algebraic quantum field theory (AQFT), which will reappear later in the talk.

Stone's theorem

Suppose we have a group of unitary operators $(U(t))_{t\in\mathbb{R}}$ on a Hilbert space $\mathcal{H}$.

Suppose that this group is strongly continuous, meaning that for each $x \in \mathcal{H}$, the map $t \mapsto U(t)x$ is continuous.

Then Stone's theorem says that there is a self-adjoint operator $H : \mathcal{D} \to \mathcal{H}$, where $\mathcal{D}$ is a dense subspace of $\mathcal{H}$, such that $U(t) = e^{-\mathrm{i}tH}$ for all t. Conversely, given such an H, $U(t) := e^{-\mathrm{i}tH}$ is a strongly continuous group of unitary operators.

Thus, the problem of constructing $(U(t))_{t\in\mathbb{R}}$ is equivalent to constructing the Hamiltonian operator H.

Appearance of probability theory

The key probabilistic idea in constructive field theory is as follows. Recall that our goal is to define H, which then generates $U(t) = e^{-\mathrm{i}tH}$ by Stone's theorem. Also recall that $\mathcal{H} = L^2(\mu)$ for some probability measure μ on the state space Ω.

Consider a stationary reversible Markov process $\{X_t\}_{t\in\mathbb{R}}$ with state space Ω and stationary distribution μ.

The process induces a Markov semigroup of symmetric linear operators $(P(t))_{t\geq 0}$ on $\mathcal{H}$ via

$$P(t)F(x) = \mathbb{E}(F(X_t)|X_0 = x).$$

(This is a semigroup because $P(s+t) = P(s)P(t)$ for $s, t \geq 0$.)

By the Hille–Yosida theorem, if the semigroup is strongly continuous, then there is a self-adjoint operator H such that $P(t) = e^{-tH}$. Key idea: Construct $\{X_t\}_{t\in\mathbb{R}}$ so that this H coincides with the Hamiltonian we want for the quantum system.

A worked example

Consider the Lorentzian action

$$S_L(\gamma) = \int_{\mathbb{R}} \left(\gamma'(t)^2 - m^2\gamma(t)^2\right) dt.$$

This is the $0+1$-dimensional free massive scalar; equivalently, it is the harmonic oscillator written in Lorentzian field-theoretic language.

The formal Lorentzian two-point function is

$$G_L(t_1, t_2) := \frac{\int \gamma(t_1)\gamma(t_2)e^{\mathrm{i}S_L(\gamma)}\,\mathcal{D}\gamma}{\int e^{\mathrm{i}S_L(\gamma)}\,\mathcal{D}\gamma}.$$

Wick rotation in this example

Write $t = -\mathrm{i}\tau$ and let $\phi(\tau) := \gamma(-\mathrm{i}\tau)$.

Then

$$dt = -\mathrm{i}\, d\tau, \qquad \gamma'(t) = \frac{d\phi}{dt} = \mathrm{i}\phi'(\tau).$$

Therefore

$$S_L(\gamma) = \mathrm{i} \int_{\mathbb{R}} \left(\phi'(\tau)^2 + m^2\phi(\tau)^2\right) d\tau = \mathrm{i} S_E(\phi),$$

where

$$S_E(\phi) := \int_{\mathbb{R}} \left(\phi'(\tau)^2 + m^2\phi(\tau)^2\right) d\tau$$

is the Euclidean action. So the Euclidean two-point function is formally

$$G_E(\tau_1, \tau_2) := \frac{\int \phi(\tau_1)\phi(\tau_2) e^{-S_E(\phi)}\, \mathcal{D}\phi}{\int e^{-S_E(\phi)}\, \mathcal{D}\phi}.$$

What probabilistic model appears?

The Euclidean theory, heuristically, is the probability measure on “all functions from $\mathbb{R}$ into $\mathbb{R}$” that has density proportional to $e^{-S_E(\phi)}$ with respect to “Lebesgue measure” on this space.

Rigorously, this is the stationary Ornstein–Uhlenbeck process

$$dX_\tau = -mX_\tau \, d\tau + \frac{1}{\sqrt{2}} \, dB_\tau.$$

Its invariant measure is the Gaussian law

$$\mu(dx) = \sqrt{\frac{2m}{\pi}} \, e^{-2mx^2} \, dx.$$

The process is Gaussian, stationary, and Markov, with covariance

$$\mathbb{E}[X_{\tau_1} X_{\tau_2}] = \frac{1}{4m} e^{-m|\tau_1 - \tau_2|}.$$

Hamiltonian for the massive theory

The OU generator on the Hilbert space $\mathcal{H} = L^2(\mu)$ is

$$Lf(x) = \frac{1}{4} f''(x) - mxf'(x).$$

Hence

$$P(\tau) = e^{\tau L} = e^{-\tau H}, \qquad Hf(x) = -\frac{1}{4} f''(x) + mxf'(x).$$

With its natural domain,

$$\mathcal{D}(H) = \Big\{ f \in L^2(\mu) : -\frac{1}{4} f'' + mxf' \in L^2(\mu) \Big\},$$

which is a dense subspace of $L^2(\mu)$.

What is $U(t)$ in this case?

Stone's theorem gives the unitary group

$$U(t) = e^{-\mathrm{i}tH}$$

on $L^2(\mu)$.

Let $q(x) = x$. Since

$$Hq = mq,$$

we get

$$U(t)q = e^{-\mathrm{i}mt}q.$$

So the coordinate observable evolves simply by multiplication by the phase $e^{-\mathrm{i}mt}$.

Lorentzian two-point function from $U(t)$

For $\tau_2 \geq \tau_1$, the Euclidean two-point function is

$$G_E(\tau_1, \tau_2) = \langle P(\tau_2 - \tau_1) q, q \rangle_{L^2(\mu)} = e^{-m(\tau_2 - \tau_1)} \langle q, q \rangle_{L^2(\mu)} = \frac{1}{4m} e^{-m(\tau_2 - \tau_1)}.$$

After Wick rotation, for $t_2 \geq t_1$,

$$G_L(t_1, t_2) = \langle U(t_2 - t_1) q, q \rangle_{L^2(\mu)} = \frac{1}{4m} e^{-\mathrm{i}m(t_2 - t_1)}.$$

Equivalently, the time-ordered Lorentzian two-point function is

$$G_L^T(t_1, t_2) = \frac{1}{4m} e^{-\mathrm{i}m|t_1 - t_2|}.$$

The Osterwalder–Schrader axioms

In Euclidean field theory, random fields are usually too rough to be evaluated pointwise. Already for a scalar field in two Euclidean dimensions, the Euclidean object is typically a random distribution, not a random function like the OU process.

Thus one should think of them as random fields that can only be tested after smearing against smooth test functions, or by averaging over small regions.

The Osterwalder–Schrader axioms are conditions on such a Euclidean theory that allow one to reconstruct the corresponding Lorentzian quantum theory.

One of the key goals of this program is to construct rigorous quantum Yang–Mills theories, which appear in the Standard Model of particle physics.

State of affairs in constructive field theory

Constructive quantum field theory has successfully produced rigorous constructions of a number of quantum field theories.

Some successful constructions

- Free scalar fields in all dimensions.
- φ^4 theory in two and three dimensions.
- Yang–Mills theories in two dimensions.
- Abelian Yang–Mills theories in two, three, and four dimensions.

Still open

- A standard-axiom construction of non-abelian Yang–Mills theory in dimension $d \geq 3$.
- A generally accepted construction of a nontrivial interacting relativistic QFT satisfying the standard axioms in four spacetime dimensions.

Why do we need a rigorous theory?

In the absence of a rigorous foundation for Yang–Mills theories, what do theoretical physicists do?

The main available tool is perturbation theory: one treats the coupling parameter g_0 as small and performs a formal power-series expansion around $g_0 = 0$.

This has been enormously successful, earning several Nobel prizes. Some aspects of perturbative calculations also admit rigorous mathematical justification, for example in work of Kevin Costello.

However, there are fundamental problems — most notably mass gap and quark confinement — that are believed to lie beyond the reach of perturbation theory. For these questions, a non-perturbative construction would be invaluable.

AQFT: the basic idea

Another rigorous approach to quantum field theory is algebraic quantum field theory (AQFT).

Instead of starting with a field at every point, AQFT assigns to each spacetime region $\mathcal{O}$ the observables that can be measured inside that region.

This collection is called a local algebra; informally, it is the algebra of quantities one can add, multiply, and take adjoints of inside $\mathcal{O}$.

If $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then every observable from the smaller region should also belong to the bigger one.

AQFT: a toy example

In our one-dimensional massive example, let $I \subseteq \mathbb{R}$ be a time interval.

Here the Hilbert space is

$$\mathcal{H} = L^2(\mu), \qquad \mu(dx) = \sqrt{\frac{2m}{\pi}} e^{-2mx^2}\, dx.$$

Define a bounded operator $W_0 : \mathcal{H} \to \mathcal{H}$ by

$$(W_0\psi)(x) = e^{\mathrm{i}x}\psi(x), \qquad \psi \in L^2(\mu).$$

If $t_0 \in I$, then

$$W(t_0) := U(t_0)^* W_0 U(t_0).$$

This is a clearly defined bounded operator on $L^2(\mu)$, and it is an element of the local algebra attached to the interval I.

AQFT and the OS framework

The two frameworks play different roles.

The Osterwalder–Schrader (OS) axioms start from Euclidean probabilistic data: a Euclidean random field, or equivalently its Euclidean correlation functions.

The AQFT picture is what one wants on the Lorentzian side: a Hilbert space, a time-evolution group, and algebras of observables attached to spacetime regions.

The connection is that OS reconstruction gives one route from the first picture to the second: if the Euclidean data satisfy the OS axioms, especially reflection positivity, then one can build the Lorentzian Hilbert space and dynamics, and from the resulting smeared observables obtain the AQFT-type net.

Yang–Mills theories

We now turn to a particularly important class of quantum field theories, known as Yang–Mills theories.

These include the gauge theories underlying electromagnetism, the weak interaction, and the strong interaction.

To see where these theories come from, let us begin with the simplest classical model: Maxwell's equations in vacuum.

Maxwell's equations in vacuum

An electromagnetic field is specified by an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$.

This means that, for each time $t \in \mathbb{R}$ and each point $x \in \mathbb{R}^3$, we have vectors $\mathbf{E}(t, x), \mathbf{B}(t, x) \in \mathbb{R}^3$.

In vacuum, Maxwell's equations govern the behavior of $(\mathbf{E}, \mathbf{B})$:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= 0 && \text{(Gauss's law for electricity)},\\
\nabla \cdot \mathbf{B} &= 0 && \text{(Gauss's law for magnetism)},\\
\nabla \times \mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} && \text{(Faraday's law of induction)},\\
\nabla \times \mathbf{B} &= \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} && \text{(Ampère's law in vacuum)},
\end{aligned}$$

Henceforth, we will work in units where $\hbar = c = 1$.

Gauge field

Can Maxwell's equations be recovered as the equations satisfied by critical points of an action?

Yes, although the route is somewhat indirect.

First, we introduce a gauge-dependent object called a gauge potential or gauge field

$$A : \mathbb{R}^4 \to (\mathrm{i}\mathbb{R})^4.$$

Let us write $A = (-\mathrm{i}\phi, \mathrm{i}\mathbf{A})$, where $\phi : \mathbb{R}^4 \to \mathbb{R}$ and $\mathbf{A} : \mathbb{R}^4 \to \mathbb{R}^3$. Then A determines $(\mathbf{E}, \mathbf{B})$ via

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

The correspondence between A and $(\mathbf{E}, \mathbf{B})$ is not one-to-one: different potentials can determine the same electric and magnetic fields. Thus A is gauge-dependent rather than directly observable.

Action of the gauge field

Henceforth, we will index elements of $\mathbb{R}^4$ as $x = (x_0, x_1, x_2, x_3)$, with x_0 denoting the time coordinate.

We define the action of $A = (A_0, A_1, A_2, A_3)$ by

$$S(A) = -\frac{1}{2g_0^2} \int_{\mathbb{R}^4} \sum_{0 \leq i < j \leq 3} \eta_i \eta_j (\partial_i A_j(x) - \partial_j A_i(x))^2 dx,$$

where $\eta_0 = 1$, $\eta_1 = \eta_2 = \eta_3 = -1$, and g_0 is a parameter known as the coupling constant.

One can show that A is a critical point of S if and only if the $(\mathbf{E}, \mathbf{B})$ obtained from A solves Maxwell's equations.

(This was discovered by Weyl in the 1920s, following contributions by Minkowski and Hilbert.)

Generalization by Yang and Mills

The action introduced by Weyl allowed the electromagnetic field to be formulated as a gauge theory, eventually leading to quantum electrodynamics through the work of Tomonaga, Schwinger, Feynman, Dyson, and others.

In the 1950s, Chen-Ning Yang and Robert Mills realized that gauge theory admits elegant non-abelian generalizations. These ideas later became central to the gauge-theoretic descriptions of the weak and strong interactions.

The first key observation is that the gauge field $A : \mathbb{R}^4 \to (\mathrm{i}\mathbb{R})^4$ in Weyl's theory can be viewed as a $\mathfrak{u}(1)$-valued 1-form

$$A = \sum_{j=0}^{3} A_j dx_j ,$$

since $\mathfrak{u}(1) = \mathrm{i}\mathbb{R}$.

Generalization by Yang and Mills, continued

The next observation is that the term $\partial_i A_j - \partial_j A_i$ appearing in Weyl's action is simply F_{ij}, where the $\mathfrak{u}(1)$-valued 2-form

$$F = \sum_{0 \leq i < j \leq 3} F_{ij} dx_i \wedge dx_j$$

is the curvature form of A.

At the level of pure gauge fields, the Yang–Mills idea is to replace the abelian Lie algebra $\mathfrak{u}(1)$ by a non-abelian Lie algebra. The Lie algebras $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ are central examples in the gauge-theoretic descriptions of the weak and strong interactions, respectively.

The Yang–Mills action

Let $\mathfrak{g}$ be any matrix Lie algebra.

Consider now a theory in which the gauge field A is a $\mathfrak{g}$-valued 1-form on $\mathbb{R}^4$. The curvature form of A is then the $\mathfrak{g}$-valued 2-form

$$F = \sum_{0 \leq i < j \leq 3} F_{ij} dx_i \wedge dx_j,$$

where

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j].$$

The Yang–Mills action of A is

$$S(A) = -\frac{1}{2g_0^2} \int_{\mathbb{R}^4} \sum_{0 \leq i < j \leq 3} \eta_i \eta_j \operatorname{Tr}(F_{ij}(x)^2) dx,$$

where $\eta_0 = 1$ and $\eta_1 = \eta_2 = \eta_3 = -1$.

The Yang–Mills existence problem

The Yang–Mills existence problem asks for a mathematically rigorous meaning of expectation values of the form

$$\langle F(A) \rangle = \frac{\int F(A) e^{iS(A)} \, DA}{\int e^{iS(A)} \, DA},$$

where $\int \cdots DA$ denotes formal integration over the space of all gauge fields.

In a gauge theory, the observable $F(A)$ should be gauge-invariant. A basic example is the Wilson loop observable associated to a closed loop ℓ:

$$W_\ell(A) = \operatorname{Tr} \mathcal{P} \exp\left(\int_\ell A \right),$$

where $\mathcal{P}$ denotes path ordering.

Thus the formal expression above hides two difficulties: the infinite-dimensional integration, and the gauge redundancy.

Lattice gauge theories

Euclidean Yang–Mills theories are the random fields, or probability measures on gauge fields modulo gauge transformations, that one wants to construct in order to recover quantum Yang–Mills theory.

Unfortunately, we do not yet know how to construct these continuum objects for four-dimensional non-abelian theories, or even in the full standard sense for three-dimensional non-abelian theories.

Lattice gauge theories are discrete approximations to Euclidean Yang–Mills theories, and they are mathematically well defined.

Bałaban’s work

In the 1980s, Tadeusz Bałaban undertook, in a long series of papers, a program to construct 3D and 4D Euclidean Yang–Mills theories as continuum limits of lattice gauge theories.

To the best of my knowledge, the full continuum construction was not completed: Bałaban proved that a crucial property known as ultraviolet stability holds, but the construction of the Euclidean Yang–Mills theories themselves was not completed.

Recent progress

Over the past decade, the probability community has devoted substantial effort to the Yang–Mills problem and related topics.

Two main strands of work have emerged:

- Gauge-string duality in lattice gauge theories.
- Stochastic quantization of Yang–Mills theories.

Gauge-string duality

Gauge-string duality is the idea that certain calculations in gauge theories can be re-expressed in terms of string-like objects, such as sums over surfaces.

This has been a major area of activity in theoretical physics for roughly the last thirty years, but it has remained largely out of the reach of rigorous mathematics.

One result in this direction, from a 2019 paper of mine, shows that calculations in certain lattice gauge theories can be transferred to certain "lattice string theories."

This is still quite far from the full continuum story, but it suggests that some rigorous form of the duality may be accessible.

In recent work, this line has seen significant generalizations and new results, including a series of papers by Sheffield and collaborators.

Stochastic quantization

Euclidean Yang–Mills theories are the random fields one wants to construct in order to recover quantum Yang–Mills theory.

Parisi and Wu proposed an indirect route: construct an auxiliary Markov process whose stationary distribution is the Euclidean Yang–Mills law, rather than constructing that law directly.

The corresponding dynamics take the form of a singular SPDE, the stochastic Yang–Mills heat flow.

Recent singular-SPDE methods now yield natural state spaces and dynamics in two dimensions, local-in-time renormalized dynamics for related three-dimensional models, and new approaches to classical constructive models such as Φ^4_3.

Quantum gravity models

Besides Yang–Mills theories, another important class of QFT models needing a rigorous framework is given by models of quantum gravity.

These models are often harder to make rigorous, because many do not fit into the standard constructive-QFT framework; in particular, the usual Osterwalder–Schrader route may fail.

Let me end with one example from forthcoming work: timelike Liouville field theory.

Forthcoming work: timelike Liouville theory

One example is timelike Liouville field theory on the cylinder $M = \mathbb{R} \times S^1$.

This is a candidate model for positive-curvature two-dimensional quantum gravity.

It is not expected to fit into the Osterwalder–Schrader (OS) framework — the usual Euclidean-to-Lorentzian reconstruction scheme — because the needed positivity is absent.

The formal Lorentzian action is

$$S(\phi) = \frac{1}{4\pi} \int_M \left(-(\partial_t \phi)^2 + (\partial_x \phi)^2 - 4\pi \mu e^{2b\phi} \right) dt\, dx.$$

After Wick rotation, the Euclidean kinetic term has the wrong sign, so there is no standard probabilistic model behind it.

Forthcoming work: correlators

The paper focuses on a special class of observables — called the integer screening sector — where the formulas remain explicit enough to analyze.

Starting from a finite-volume regularization, together with the renormalizations needed to make the formulas finite, it constructs Euclidean correlation functions on the cylinder.

These are then analytically continued in the time variables to produce Lorentzian correlation functions, written as explicit contour integrals.

One consequence is locality: observables at spacelike separated points commute, as relativity suggests.

Forthcoming work: AQFT without positivity

To avoid singularities at individual points, one averages observables over small regions; these are the smeared observables.

For these observables, the Lorentzian correlators lead to an AQFT-type description — that is, an algebraic description of which observables belong to which spacetime regions.

One gets a mathematically controlled space of states and observables, with translations acting naturally, and with commuting observables in spacelike separated regions (locality).

The missing ingredient is positivity. Thus, this is not the usual Hilbert space version of AQFT, even though it retains an AQFT-type local net structure.

Conclusion

A probabilistically defined quantum field theory is, roughly speaking, a Lorentzian quantum theory reconstructed from suitable Euclidean random data.

For Yang–Mills theory, constructing the relevant Euclidean object in four dimensions remains a central open problem; this is one reason the Clay problem is so hard.

The final example suggests that not every interesting Lorentzian theory is likely to come from the usual Osterwalder–Schrader route. But even there, probability can enter in an indirect way.

It is an exciting time to work at the intersection of probability and quantum field theory.

Thank you!