CFT Perspectives on 2D Percolation

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Simons Foundation, 05/01/2026

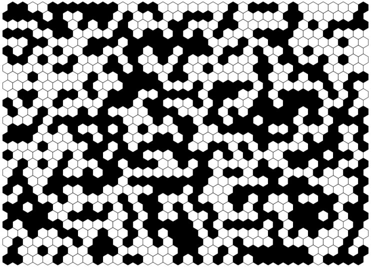
Main message

Critical 2D percolation is governed by an exactly solvable conformal field theory.

Perfect playground where probability meets CFT.

Outline

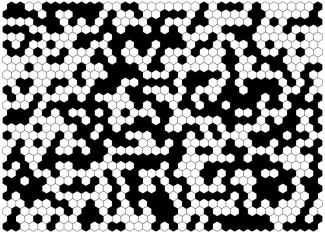
1. 2D Bernoulli percolation and conformal invariance.
2. Three classical CFT perspectives on 2D percolation.
3. Probablistic approaches to exact solvability.
4. Current frontier: towards conformal bootstrap.

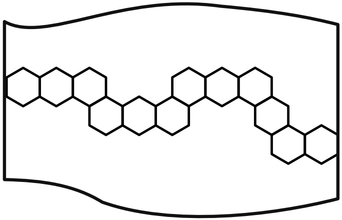


- Bernoulli site percolation on triangular lattice: color each site white (open) with probability p

- Observables: connectivity property e.g. crossing event for topological quadrangles.

- Phase transition at $p_c = 1/2$
  $p < p_c$: macroscopic connectivity probability $\to 0$
  $p = p_c$: non-trivial continuum limit.
  $p > p_c$: macroscopic connectivity probability $\to 1$.





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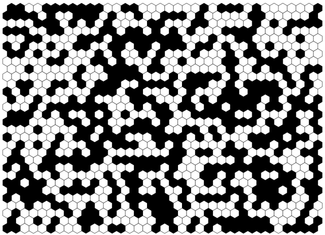

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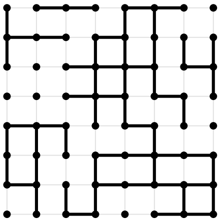
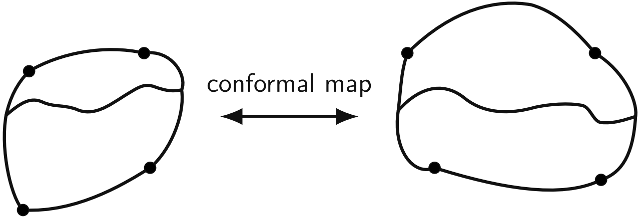


- Universality:
  continuum limit independent of lattice details
  (e.g. bond percolation on square lattice)

Conformal Symmetry of Continuum Limit

Aizenman ($\sim$ 1990)

Quadrangle crossing probabilities are conformally invariant.



Smirnov (2001)

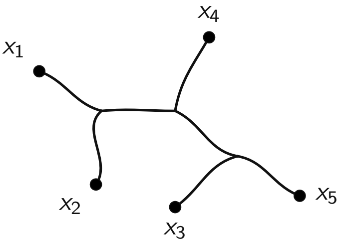
Existence and conformal invariance of continuum limit for site percolation on triangular lattice.

Conformal Symmetry of Continuum Limit

$G_n(x_1, x_2, \ldots, x_n) := \lim_{\delta \to 0} \delta^{-n\Delta_1} \mathbb{P}[x_1^\delta, x_2^\delta, \ldots, x_n^\delta$ are connected] exists for some Δ_1.

Conformal covariance of G_n — Aizenman (1995)

$G_n(x_1, x_2, \ldots, x_n) = \prod_{i=1}^n |\psi'(x_i)|^{\Delta_1} G_n(\psi(x_1), \psi(x_2), \ldots, \psi(x_n))$.



Proved for site percolation on triangular lattice by Garban–Pete–Schramm (2013), Camia (2024).

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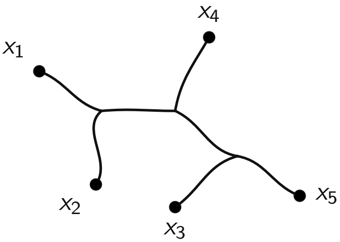

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Corollary

- $G_3(x_1, x_2, x_3) = C_3 |x_1 - x_2|^{-\Delta_1} |x_1 - x_3|^{-\Delta_1} |x_2 - x_3|^{-\Delta_1}$.
- $G_2(x_1, x_2) = C_2 |x_1 - x_2|^{-2\Delta_1}$.

Quadrangle crossing probability: Cardy (1992); Smirnov (2001)

$\mathbb{P}\big[\text{left-right crossing of quadrangle with cross ratio } x\big] = \frac{\Gamma(2/3)}{\Gamma(1/3)\Gamma(4/3)}\, x^{1/3}\, {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; x\right)$

Exact Solvability from Conformal Field Theory

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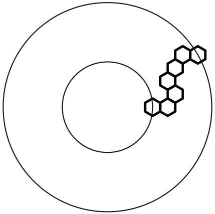
One-arm exponent: Nienhuis (1984); Lawler–Schramm–Werner (2002)

$\Delta_1 = 5/48$.

$\mathbb{P}[\text{annulus crossing}] = (\frac{r}{R})^{\Delta_1 + o(1)}$ as $r/R \to 0$.



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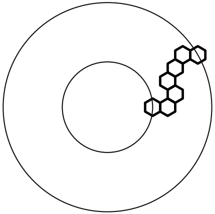

3-point connectivity constant: Delfino–Viti (2010); Ang–Cai–S.–Wu (2024)

$R = \frac{G_3(x_1,x_2,x_3)}{\sqrt{G_2(x_1,x_2)G_2(x_1,x_3)G_2(x_2,x_3)}} = \mathrm{ImDOZZ}_{c=0}(\Delta_1, \Delta_1, \Delta_1) = 1.02201\ldots$

All these exact results are predicted in physics using CFT ideas.

Belavin–Polyakov–Zamolodchikov (1984)

Operator-algebra framework of 2D CFT: Virasoro algebra + OPE $\Longrightarrow$ conformal bootstrap.

- Correlation functions with degenerate fields satisfy (BPZ) differential equations.
- Minimal models: a series of exactly solvable CFTs.
- Local observables of certain 2D statistical physics models are governed by minimal models.
- E.g. spin and energy correlations for Ising and 3-state Potts models.

2D statistical physics and conformal field theory

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It is surprising (to me) that percolation can be exactly solved by CFT.

- Percolation observables (connectivity) are highly non-local, so why described by a CFT?
- Why is this CFT exactly solvable? ($c = 0$ minimal model is trivial.)

Three classical CFT perspectives on 2D percolation

Coupling with q-state Potts model

- Map to q-state Potts model.
- Analyze via minimal-model CFT and analytically continue to $q = 1$.

Coulomb gas method

- Map to height function.
- Height function converges to Gaussian free field.

Transfer matrix

- Transfer matrix expressed via Temperley–Lieb algebra.
- Temperley–Lieb algebra converges to Virasoro algebra.

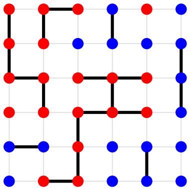
q-state Potts model (q integer)

$\sigma \in \{1, 2, \ldots, q\}^V$, $\qquad H(\sigma) = \sum_{(u,v)} \mathbb{1}\{\sigma_u = \sigma_v\}$, $\qquad Z = \sum_\sigma \exp(-\beta H(\sigma))$.

Ising model = 2-state Potts model

Fortuin–Kasteleyn q-random-cluster model

- configuration weight: $p^{\#\mathrm{open}}(1-p)^{\#\mathrm{closed}} q^{\#\mathrm{clusters}}$.
- q vary continuously. $\qquad p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$.
- $q = 1$ case $\Longrightarrow$ Bernoulli bond percolation.



q-random-cluster model vs q-state Potts model

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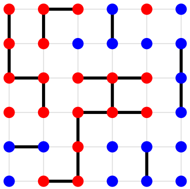


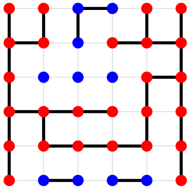
Edwards–Sokal coupling: $p = 1 - e^{-\beta}$

- $\mathbb{P}_{\text{Potts}}[\sigma_x = \sigma_y] - q^{-1} = \left(1 - q^{-1}\right) \mathbb{P}_{\text{random-cluster}}[x \text{ and } y \text{ are connected}]$.
- More generally, ***q*-state Potts spin-spin correlations are expressed by connectivity.**

Consequence of random-cluster/Potts coupling

$\mathbb{P}[\text{quad crossing}] = Z_{\alpha\alpha} - Z_{\alpha\beta}$. $\alpha \neq \beta \in \{1, 2, \ldots, q\}$.

$Z_{\alpha\beta}$: partition function with (α, free, β, free) boundary condition.



- For $q \in \{2, 3\}$, $Z_{\alpha\alpha}$ and $Z_{\alpha\beta}$ are 4-point boundary correlation functions for minimal models.
- Satisfy BPZ-type differential equations (hypergeometric equation).
- Analytic continuation to $q \to 1$ gives percolation crossing probability.

Cardy's argument for his quad-crossing formula

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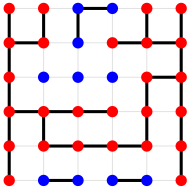

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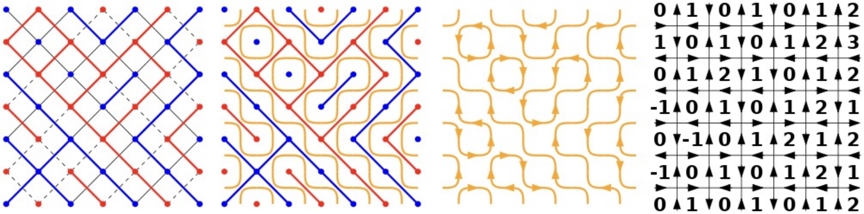
Baxter–Kelland–Wu loop/height correspondence

clusters → loops → oriented loops → height field



From Duminil-Copin, Gagnebin, Harel, Manolescu, Tassion (2021)

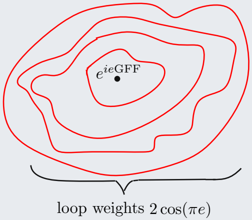
Nienhuis (1984) for $q \in (0,4]$; Duminil-Copin–Kozlowski–Lammers–Manolescu (2026) for $q \in [1,4]$

Height function converges to the Gaussian free field with variance $2/\arccos(-\sqrt{q}/2)$.

Derivation of one-arm exponent

Nienhius (1984)

- Planarity: 2 points are connected $\iff$ no loops in between.
- change loop weight around a point to $2\cos(\pi e) \iff$ Add electric operator insertion $e^{ie\mathrm{GFF}}$.



- no-loop condition $\implies$ loop weight $2\cos(\pi e) = 0 \implies e = 1/2$.

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- In Coulomb gas framework, $e^{ie\mathrm{GFF}}$ has scaling dimension

$$\frac{e^2 - (1-g)^2}{2g}, \qquad \text{with} \qquad g = \frac{1}{\pi / \arccos(-\sqrt{q}/2)}.$$

- One-arm exponent $\Delta_1 = (1 - 4(1-g)^2)/(8g)$.
- Bond percolation: $q = 1 \implies g = \frac{2}{3} \implies \Delta_1 = \frac{5}{48}$.

Ongoing work by Chen–Duminil-Copin–He–Krachun–Manolescu–Xia for $q \in [1, 4]$.

Transfer matrix and Temperley–Lieb algebra

- Transfer matrix T: keep track of loop configuration slice by slice in the q-random-cluster model.

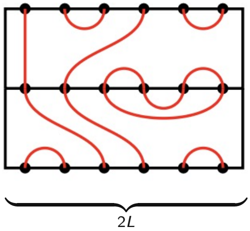
- Local join/split operations are generated by Temperley–Lieb algebra with $n = \sqrt{q}$:

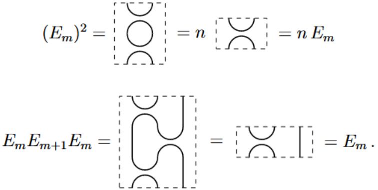
$$E_m^2 = nE_m, \qquad E_m E_{m\pm1} E_m = E_m.$$

- $T = \frac{q^{L/2}}{(1+\sqrt{q})^{2L-1}} \left(\prod_{m=1}^{L}(1 + E_{2m-1})\right)\left(\prod_{m=1}^{L-1}(1 + E_{2m})\right)$.

- Partition function $Z = \mathrm{Tr}(T^M)$.

- Related to quantum group $U_q(sl_2)$; Yang–Baxter.





From Jesper Lykke Jacobsen

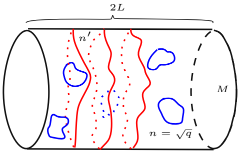
Conjecture Saleur–Bauer (1989); Koo–Saleur (1993); Folklore ...

Temperley–Lieb algebra $\Longrightarrow$ Virasoro algebra.

Saleur–Bauer (1989); Dubail–Jacobsen–Saleur (2010)

$Z_{L,M}(n,n') = \sum_{j=0}^{L} \frac{\sin(2j+1)\chi'}{\sin\chi'} \mathrm{tr}_{\mathcal{V}_{2j}^L} T^M,$

- $\mathcal{V}_j^L$: rep. space of length-$(2L)$ diagrams with j strings.
- contractible loop weight: $n = \sqrt{q}$. (percolation: $n = 1$.)
- non-contractible loop weight: $n' = 2\cos\pi\chi'$.



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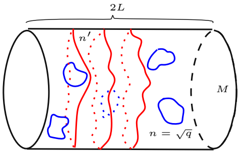


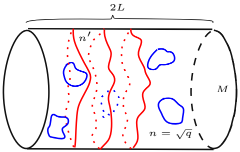
Conjecture $\Longrightarrow$ $\lim \operatorname{tr}_{\mathcal{V}_{2j}^L} T^M = K_{1,1+j}$ as $L, M \to \infty$ and $L/M = \tau$.

- $K_{1,1+j}(\mathbf{q})$: Virasoro character on Kac module with $\mathbf{q} = e^{-\pi/\tau}$.
- $K_{1,1+j}(\mathbf{q}) = \mathbf{q}^{-\frac{\mathbf{c}}{24}}(\prod_{k=1}^{\infty}(1-\mathbf{q}^k))^{-1}(\mathbf{q}^{\frac{gj^2}{4}-\frac{(1-g)j}{2}} - \mathbf{q}^{\frac{g(j+2)^2}{4}+\frac{(1-g)(j+2)}{2}})$.

Loop model for percolation on the annulus

Generating function for non-contractible loops

- $\lim Z_{L,M}(n, n') = Z(n, n')$ as $L, M \to \infty$ and $L/M = \tau$.
- $Z_{q=1}(n, n') = \sum_{j=0}^{\infty} \frac{\sin(2j+1)\chi'}{\sin \chi'} K_{1,1+j}(\mathbf{q})$.



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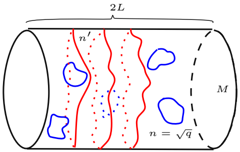
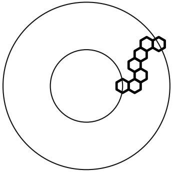


Annulus crossing formula Cardy (2006); S.–Xu–Zhuang (2024)

$$\mathbb{P}[\text{annulus crossing}] = \frac{Z_{q=1}(n,0)}{Z_{q=1}(n,n)} = \sqrt{\frac{3}{2}} \cdot \frac{\eta(6i\tau)\eta(\frac{3}{2}i\tau)}{\eta(2i\tau)\eta(3i\tau)}$$

$$\tau = \frac{1}{2\pi}\log(R/r), \quad \eta(z) = e^{i\pi z/12}\prod_{n=1}^{\infty}(1 - e^{2ni\pi z}).$$



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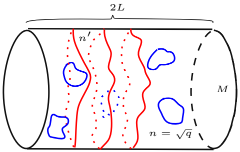

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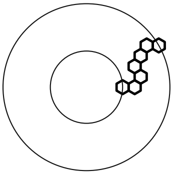

Li–Liu–S.–Zhuang (2026+) based on Ang–Remy–S. (2025)

We have $\lim \mathrm{tr}_{\mathcal{V}_{2j}^L} T^M = K_{1,1+j}$ assuming scaling limits of bond percolation models on $\mathbb{Z}^2$.

Probabilistic approaches to exact solvability

Global strategy

- Scaling limit of 2D critical percolation is encoded by SLE_6 curves.
- Exact solvability of $\mathrm{SLE}_6 \implies$ exact solvability of percolation.

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Loewner evolution

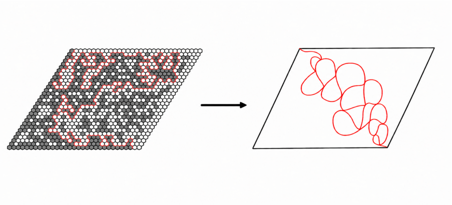
- SLE $=$ Loewner evolution with Brownian motion as the driving function.
- Martingale observables, Itô calculus, 2nd order differential equation.

SLE coupled with Liouville quantum gravity

- Mating of trees.
- Exact solvability of Liouville conformal field theory.

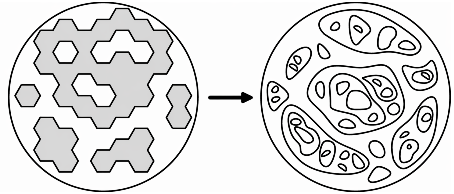
Remark: so far almost orthogonal to the three CFT perspectives.

- Scaling limit of interfaces in 2D critical lattice models should be SLE_κ curves with $\kappa > 0$. Schramm (1999).



- Scaling limit of the entire loop collection is described by Conformal Loop Ensemble (CLE_κ). Sheffield (2009), Werner–Sheffield (2012).

- Percolation corresponds to SLE_6 and CLE_6. Proved for site percolation on the triangular lattice. Smirnov (2001), Camia–Newman (2006).

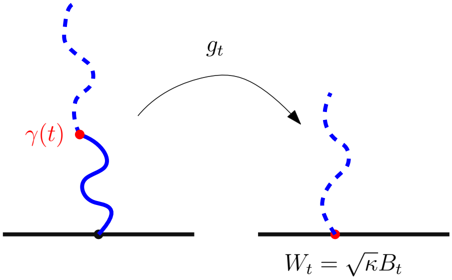


SLE as a Loewner Evolution

SLE on $\mathbb{H}$ from 0 to ∞

$$dg_t(z) = \frac{2}{g_t(z) - W_t}\, dt.$$

- $g_t(z) = z + \frac{2t}{z} + O(\frac{1}{|z|^2})$ as $z \to \infty$.
- $W_t = \sqrt{\kappa}\, B_t$. B_t: Brownian motion.
- g_t: Loewner map. W_t: driving function.



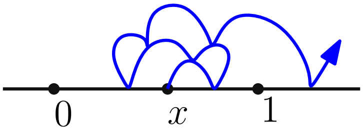
Recipe to exact solvablity

Identify martingale observable $\Longrightarrow$ 2nd order differential equation (BPZ type).

SLE derivation of Cardy's crossing formula

SLE formulation of crossing event

- γ: SLE_6 curve on $\mathbb{H}$ from x to ∞.
- γ hits $[1,\infty)$ before $(-\infty,0]$
  $\Longleftrightarrow$ crossing from $[0,x]$ to $[1,\infty)$ occurs.



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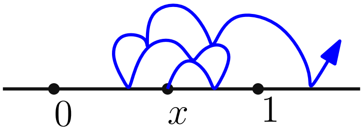
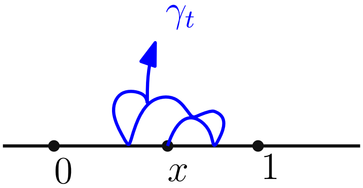


Martingale observable

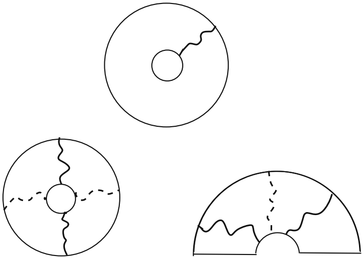
- Z_t: cross ratio of $(0,\gamma_t,1,\infty)$ as in $\mathbb{H}\setminus\gamma[0,t]$.
- $f(x) := \mathbb{P}_x\big[\gamma \text{ hits } [1,\infty) \text{ before } (-\infty,0]\big]$.
- $f(Z_t)$ is a martingale.



- Itô's formula $\implies$ $3z(1-z)f''(z)+2(1-2z)f'(z)=0$.
- $f(0)=0$ and $f(1)=1$ $\implies$ Cardy's formula.

Scaling exponent/fractal dimension

- One-arm exponent $\Delta_1 = \frac{5}{48}$.
  realized as the leading eigenvalue of a BPZ type differential operator.
- Polychromatic k-arm exponents:
  bulk: $\frac{k^2-1}{12}$ boundary: $\frac{k(k+1)}{6}$.



Exact results from the Loewner evolution approach

Scaling exponent/fractal dimension

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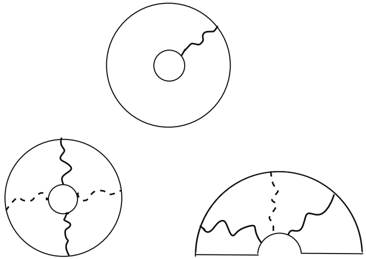
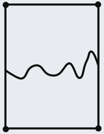
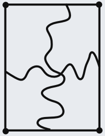
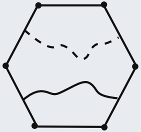


Connectivity probability on planar domain with boundary marked points


Cardy


Watts


Schramm


multi-crossing

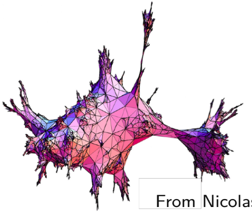
Lawler–Schramm–Werner, **Bauer–Bernard**–Kytölä, Dubédat, Zhan, Beliaev–Viklund, Peltola–Wu, ...

- a theory for random surface with conformal geometry: $e^{\text{conformal factor}}(d^2x + d^2y)$.
- conformal factor sampled from Liouville conformal field theory (central charge c_L)

LQG describes scaling limit of random triangulations under conformal embedding

- Uniformly sampled triangulation $\implies c_L = 26$.
- Coupled with conformal matter $\implies c_m + c_L = 26$. c_m: matter central charge.

Uniform triangulation

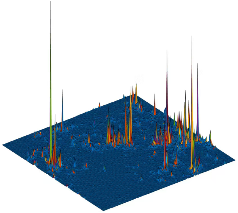


From Nicolas Curien

conformal embedding

scaling limit

Liouville conformal field theory



Knizhnik–Polyakov–Zamolodchikov Relation

- $d_{\text{Euclidean}} = \text{KPZ}(d_{\text{quantum}})$. d: dimension of a planar fractal.
- $\text{KPZ}(x) = \frac{5}{3}x - \frac{1}{3}x^2$ for pure gravity ($c_L = 26$).

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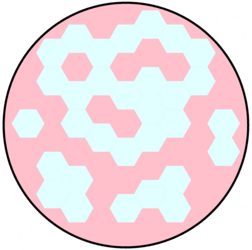
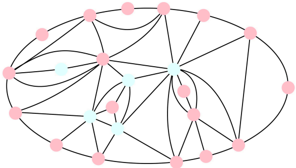
KPZ derivation of $\Delta_1 = \frac{5}{48}$

- for a lattice region with n^2 vertices,
  the size of boundary connecting cluster is $\sim n^{2-\Delta_1}$.
- On random triangulation of same size, the answer is $n^{7/4}$.
- $d_{\text{Euclidean}} = 2 - \Delta_1 = \text{KPZ}(\frac{7}{4}) = \frac{91}{48} \implies \Delta_1 = \frac{5}{48}$.





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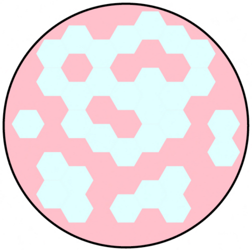
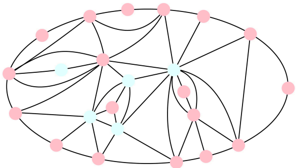
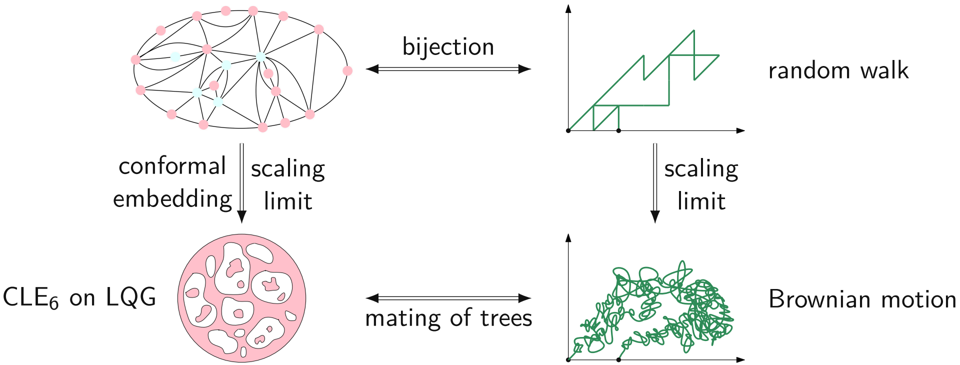




- Derived from the CFT description of LQG. (KPZ '88)
- Justified by comparing to the discrete description.
- Powerful framework to study random fractals. (Duplantier)



Sheffield (2010), Duplantier–Sheffield (2011), Duplantier–Miller–Sheffield (2014), ...

- Turn KPZ derivation of scaling exponents into rigorous proof.
- Does not give correlation function/partition function information.

Overall strategy

- Conformal factors of LQG surfaces are described by Liouville CFT.
- Mating of trees gives exact solvability of SLE/CLE coupled with Liouville CFT.
- Liouville CFT itself is exactly solvable.
- Combining these two frameworks gives solvability of SLE/CLE.

Synergy between Mating of Trees and Liouville Conformal Field Theory

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Exact solvability of Liouville CFT in probabilistic framework

- DOZZ formula. Kupiainen–Rhodes–Vargas (2017)
- Conformal bootstrap. Guillarmou–KRV (2024, 2025)
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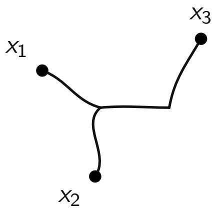
- Synergy between two frameworks was initiated by Ang, Holden, Remy, S. (2020–2022)
- Recently applied to obtain exact solvability of percolation by S. et al. (2023-present)

$$G_n(x_1, x_2, \ldots, x_n) := \lim_{\delta \to 0} \delta^{-n\Delta_1} \mathbb{P}[x_1^\delta, x_2^\delta, \ldots, x_n^\delta \text{ are connected}].$$

$$R = \frac{G_3(x_1,x_2,x_3)}{\sqrt{G_2(x_1,x_2)G_2(x_1,x_3)G_2(x_2,x_3)}}. \qquad \Delta_1 = \frac{5}{48}.$$



Delfino–Viti (2010); Ang–Cai–S.–Wu (2024)

$R = \mathrm{ImDOZZ}_{c=0}(\Delta_1, \Delta_1, \Delta_1) = 1.02201\ldots.$

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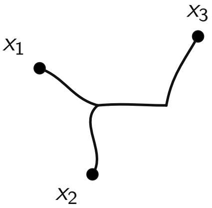

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Continuum limit of 3-point-connected percolation on random triangulation of the sphere

- **Approach 1**: CLE_6+LQG in the mating-of-trees framework; exactly solvable.
- **Approach 2**: matter CFT $\times$ Liouville CFT. (CFT description of LQG)
  - R is the matter 3-point correlation (properly normalized).
  - Liouville 3-point correlation $= \mathrm{DOZZ}_{c_\mathrm{L}}(\Delta^{\mathrm{Liou}}, \Delta^{\mathrm{Liou}}, \Delta^{\mathrm{Liou}})$.
  - $c_\mathrm{L} = 26$ and $\Delta_1 = \frac{5}{48} = \mathrm{KPZ}(\Delta^{\mathrm{Liou}})$.

3-point connectivity and the imaginary DOZZ formula

Derivation of R

(Matter 3-point correlation $\times$ Liouville 3-point correlation) is exactly solvable from mating-of-trees.

- Matter 3-point correlation (properly normalized) $= R$.
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Zamolodchikov (2005)

$\mathrm{ImDOZZ}_{c_\mathrm{M}}(\Delta_1^{\mathsf{matter}}, \Delta_2^{\mathsf{matter}}, \Delta_3^{\mathsf{matter}}) \times \mathrm{DOZZ}_{c_\mathrm{L}}(\Delta_1^{\mathsf{Liou}}, \Delta_2^{\mathsf{Liou}}, \Delta_3^{\mathsf{Liou}}) = 1.$ (up to normalization)

$c_\mathrm{M} + c_\mathrm{L} = 26$ and $\Delta^{\mathsf{matter}} = \mathsf{KPZ}(\Delta^{\mathsf{Liou}})$.

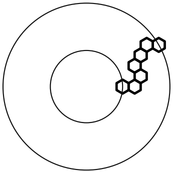
- percolation: $c_\mathrm{M} = 0$ and $c_\mathrm{L} = 26$.
- $\Delta_1^{\mathsf{matter}} = \Delta_2^{\mathsf{matter}} = \Delta_3^{\mathsf{matter}} = \Delta_1 = \frac{5}{48}$.
- $\implies R = \mathrm{ImDOZZ}_{c=0}(\Delta_1, \Delta_1, \Delta_1)$.

Annulus crossing and random moduli of LQG surfaces

Cardy (2006); S.–Xu–Zhuang (2024)

$$\mathbb{P}[\text{annulus crossing}] = \sqrt{\frac{3}{2}} \cdot \frac{\eta(6i\tau)\eta(\frac{3}{2}i\tau)}{\eta(2i\tau)\eta(3i\tau)}$$

$$\tau = \frac{1}{2\pi}\log(R/r), \quad \eta(z) = e^{i\pi z/12}\prod_{n=1}^{\infty}(1 - e^{2ni\pi z}).$$



Continuum limit of percolation with crossing on random triangulation of the annulus

- **Approach 1**: CLE_6 + LQG in the mating-of-trees framework; exactly solvable.
- **Approach 2**: matter CFT × Liouville CFT × ghost CFT.
  - $\int_0^\infty$ (Liouville theory on annulus of modulus τ) $\times Z_{\text{matter}}(\tau) Z_{\text{ghost}}(\tau)\, \mathrm{d}\tau$
  - $Z_{\text{matter}}(\tau) = \mathbb{P}[\text{annulus crossing}]$ $\qquad Z_{\text{ghost}}(\tau) = \eta(2i\tau)^2.$

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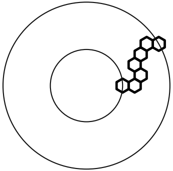

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  - $Z_{\text{matter}}(\tau) = \mathbb{P}[\text{annulus crossing}]$ $\qquad Z_{\text{ghost}}(\tau) = \eta(2i\tau)^2$.

- Ang–Remy–S. (2025): general method to derive random modulus of a random annulus in LQG.
- Apply to random annulus with or without crossing event; compare to get $Z_{\text{matter}}(\tau)$.

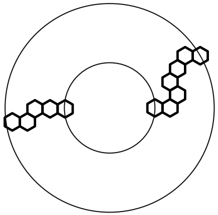
$\mathbb{P}$[annulus crossing with two disjoint open arms] $= (\frac{r}{R})^{\beta_2+o(1)}$.

Nolin–Qian–S.–Zhuang (2023)

β_2 is the unique solution in the interval $(\frac{1}{4}, \frac{2}{3})$ to the equation

$$\frac{\sqrt{36x+3}}{4} + \sin\left(\frac{2\pi\sqrt{12x+1}}{3}\right) = 0.$$

β_2 is a transcendental number. $\beta_2 = 0.35666683671288\cdots$



Beyond verifying predictions: backbone exponent

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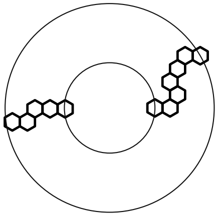


- No predictions previously other than a wrong guess: $\frac{17}{48}$.
- KPZ derivation for $\Delta_1 = \frac{5}{48}$ is hard to made work, $(\mathrm{KPZ}^{-1}(\beta_2)$ is transcendental).
- Our proof relies on the disk analog of DOZZ formula for Liouville CFT.

Towards conformal bootstrap for 2D percolation

Conformal bootstrap

- spectrum resolution of Hamiltonian via Virasoro action
- operator product expansion (OPE)

Development in physics

- method: axiomatic bootstrap + transfer matrix.
- exact predictions: 3-point function on sphere; 2-point function on disk.

Update: 1-point torus (Jesper Jacobsen's talk on Monday)

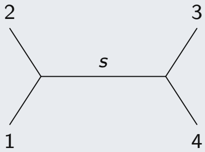
Development in mathematics

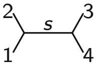
- mating of trees + Liouville CFT solves 3-point functions beyond ImDOZZ.
- method based on path integral for SLE reproduces 2-point disk prediction.

Conformal bootstrap: general framework

Conformal bootstrap expansion for 4-point sphere

$$\langle V_{\Delta_1}(z) V_{\Delta_2}(0) V_{\Delta_3}(\infty) V_{\Delta_4}(1)\rangle = \sum_{\Delta_s \in \mathcal{S}} C_{12s} C_{s34}$$



- $\mathcal{S}$: spectrum.
- C_{ijk}: structure constant.
- 

: conformal block, determined by the Virasoro algebra.

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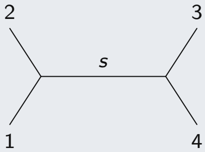

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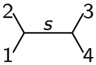
- 

: conformal block, determined by the Virasoro algebra.

- each expansion corresponds to a way of cutting a 4-punctured sphere into two 2-punctured disks.
- crossing symmetry: consistency between different expansions.

A Modern CFT Approach to 2D Percolation in Physics

Jacobsen, Nivesvivat, Ribault, Saleur et.al.

- Assume percolation (more generally loop/cluster models) satisfy conformal bootstrap.
- Annulus/torus partition functions constrain the spectrum.
- Crossing symmetry constrains structure constants.
  - further constraint by assuming degenerate fields.
  - further constraint from Temperley–Lieb algebra and variants.

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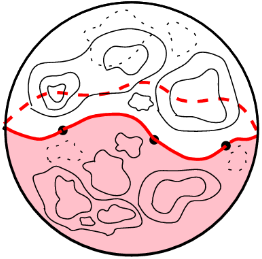
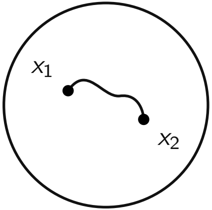
3-point function on sphere Jacobsen–Nivesvivat–Ribault-Roux (2025)

$$C_{123} = \prod_{\epsilon_1,\epsilon_2,\epsilon_3=\pm} \Gamma_\beta^{-1}\left(\frac{\beta+\beta^{-1}}{2} + \frac{\beta}{2}\left|\sum_i \epsilon_i r_i\right| + \frac{\beta^{-1}}{2}\sum_i \epsilon_i s_i\right)$$

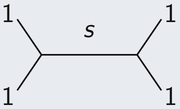


- include imaginary DOZZ but encode much more observables.
- e.g. 3 points on the same loop.

$G_{\mathrm{disk}}(x_1, x_2) := \lim_{\delta \to 0} \delta^{-2\Delta_1} \mathbb{P}[x_1^\delta \text{ and } x_2^\delta$ are connected in disk].



Downing–Jacobsen–Nivesvivat–Ribault–Saleur (2026)

$G_{\mathrm{disk}}(x_1, x_2) = \sum_{\Delta_s \in \mathcal{S}} C_{11s} R_s$ 

- $\mathcal{S} = \{(1, 2k+1) : k \in \mathbb{Z}_+\}$ as in Kac table.
- C_{11s}: sphere 3-point function.

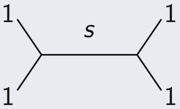
$R_{(1,2k+1)} = (-1)^{k+1}$: disk 1-point function.

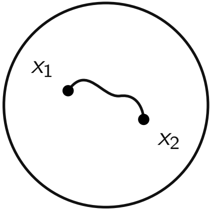
A Modern CFT Approach to 2D Percolation in Physics

$G_{\text{disk}}(x_1, x_2) := \lim_{\delta \to 0} \delta^{-2\Delta_1} \mathbb{P}[x_1^\delta \text{ and } x_2^\delta$ are connected in disk].

Downing–Jacobsen–Nivesvivat–Ribault–Saleur (2026)

$G_{\text{disk}}(x_1, x_2) = \sum_{\Delta_s \in \mathcal{S}} C_{11s} R_s$





- $\mathcal{S} = \{(1, 2k+1) : k \in \mathbb{Z}_+\}$ as in Kac table.
- C_{11s}: sphere 3-point function.

$R_{(1,2k+1)} = (-1)^{k+1}$: disk 1-point function.

Interaction with the probabilistic approach

- This framework is not fully analytic, involves guess-and-check via numercial methods.
- For the C_{123} formula, probabilistic derivation of the 3-point loop case played a crucial role in navigating the guess-and-check progress.

Path integral approach to SLE

View SLE as a measure of the form $e^{-\mathrm{Action}[\gamma]} D\gamma$.

- conformal restriction. Lawler–Schramm–Werner, Kontsevich–Suhov, Dubédat, Zhan
- large deviation, classical geometry. Wang, Viklund–Wang, Takhtajan–Teo.
- integration by parts $\Longrightarrow$ Virasoro action. Baverez–Jego; Gordina–Qian–Wang

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A probabilistic argument for 2-point disk correlations Jin–S.–Wu (2026+)

- based on the path integral approach to SLE.
- not rigorous, but there is no guess-and-check.
  - rigorous path integral so far only for SLE_κ with $\kappa \in (0, 4]$.
  - we adopt it to percolation and analytically continue to SLE_6.
- Can solve 2-point disk functions that have not been predicted in physics.

Conclusion and Outlook

- Significant progress in physics and mathematics suggest that 2D critical percolation is indeed governed by an exactly solvable CFT.
- Complete understanding has not been achieved. (n-point sphere is not pinned down.)
- The next step is to combine all methods to reach a physically complete picture.

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Rigorous understanding of CFT perspectives remains a major mathematical challenge.

- connection to minimal model?
- rigorous Coulomb gas and connection to imaginary Liouville theory?
- connection to Yang–Baxter/quantum group integrability from Temperley–Lieb algebra?
- rigorous path integral framework for SLE_6 and CLE_6?
- deeper understanding of $c < 1$ conformal block and log CFT?
- Where does backbone exponent fit in? monochromatic k-arm with $k \geq 3$?

Thank you!

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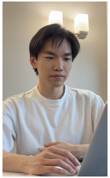

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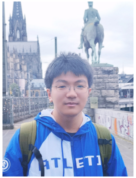

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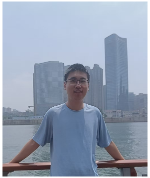

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