

CFT Perspectives on 2D Percolation

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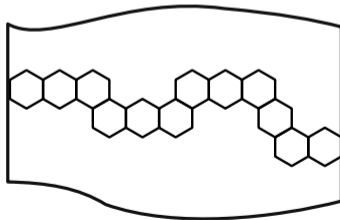
Critical 2D percolation is governed by an exactly solvable conformal field theory.

Perfect playground where probability meets CFT.

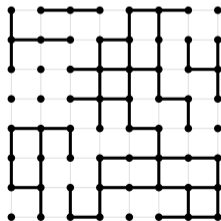
Outline

- 1 2D Bernoulli percolation and conformal invariance.
- 2 Three classical CFT perspectives on 2D percolation.
- 3 Probabilistic approaches to exact solvability.
- 4 Current frontier: towards conformal bootstrap.

- Bernoulli site percolation on triangular lattice: color each site white (open) with probability p
- Observables: connectivity property
e.g. crossing event for topological quadrangles.
- **Phase transition** at $p_c = 1/2$
 $p < p_c$: macroscopic connectivity probability 0
 $p = p_c$: **non-trivial continuum limit.**
 $p > p_c$: macroscopic connectivity probability 1.

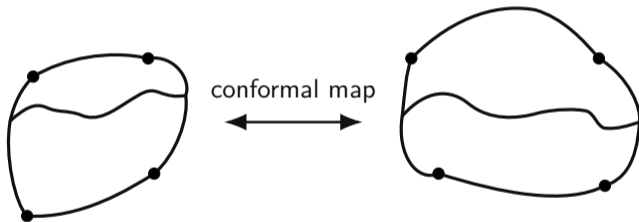


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 $p = p_c$: **non-trivial continuum limit.**
 $p > p_c$: macroscopic connectivity probability 1.
- **Universality:**
continuum limit independent of lattice details
(e.g. bond percolation on square lattice)



Aizenman (1990)

Quadrangle crossing probabilities are conformally invariant.



Smirnov (2001)

Existence and conformal invariance of continuum limit for site percolation on triangular lattice.

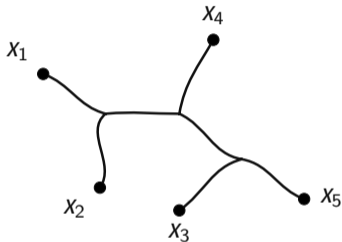
$G_n(x_1, x_2, \dots, x_n) := \lim_0^{-n\Delta_1} \mathbb{P}[x_1, x_2, \dots, x_n \text{ are connected}]$ exists for some Δ_1 .

Conformal covariance of G_n

Aizenman (1995)

$$G_n(x_1, x_2, \dots, x_n) = \prod_{i=1}^n |x_i|^{\Delta_1} G_n(|x_1|, |x_2|, \dots, |x_n|).$$

Proved for site percolation on triangular lattice by
Garban–Pete–Schramm (2013), Camia (2024).

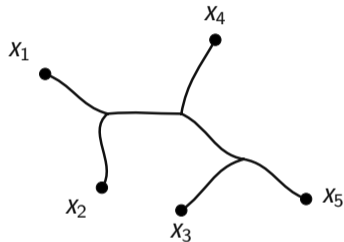


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Corollary

- $G_3(x_1, x_2, x_3) = C_3 |x_1 - x_2|^{-\Delta_1} |x_1 - x_3|^{-\Delta_1} |x_2 - x_3|^{-\Delta_1}$.
- $G_2(x_1, x_2) = C_2 |x_1 - x_2|^{-2\Delta_1}$.

Quadrangle crossing probability: Cardy (1992); Smirnov (2001)

P left-right crossing of quadrangle with cross ratio $x = \frac{\Gamma(2/3)}{\Gamma(1/3)\Gamma(4/3)} x^{1/3} {}_2F_1\left(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; x\right)$

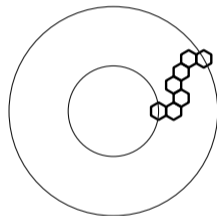
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One-arm exponent: Nienhuis (1984); Lawler–Schramm–Werner (2002)

$\Delta_1 = 5/48$.

P[annulus crossing] = $(\frac{r}{R})^{\Delta_1 + o(1)}$ as $r/R \rightarrow 0$.



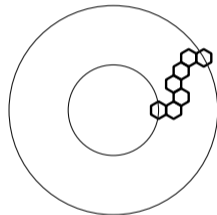
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3-point connectivity constant: Delfino–Viti (2010); Ang–Cai–S.–Wu (2024)

$$R = \frac{G_3(x_1, x_2, x_3)}{G_2(x_1, x_2)G_2(x_1, x_3)G_2(x_2, x_3)} = \text{ImDOZZ}_{c=0}(\Delta_1, \Delta_1, \Delta_1) = 1.02201\dots$$

All these exact results are predicted in physics using CFT ideas.

Belavin–Polyakov–Zamolodchikov (1984)

Operator-algebra framework of 2D CFT: Virasoro algebra + OPE = conformal bootstrap.

- Correlation functions with degenerate fields satisfy (BPZ) differential equations.
- Minimal models: a series of exactly solvable CFTs.
- Local observables of certain 2D statistical physics models are governed by minimal models.
- E.g. spin and energy correlations for Ising and 3-state Potts models.

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It is surprising (to me) that percolation can be exactly solved by CFT.

- Percolation observables (connectivity) are highly non-local, so why described by a CFT?
- Why is this CFT exactly solvable? ($c = 0$ minimal model is trivial.)

Coupling with q -state Potts model

- Map to q -state Potts model.
- Analyze via minimal-model CFT and analytically continue to $q = 1$.

Coulomb gas method

- Map to height function.
- Height function converges to Gaussian free field.

Transfer matrix

- Transfer matrix expressed via Temperley–Lieb algebra.
- Temperley–Lieb algebra converges to Virasoro algebra.

q -state Potts model (q integer)

$$\{1, 2, \dots, q\}^V, \quad H(\sigma) = \sum_{(u,v)} \{ \sigma_u = \sigma_v \}, \quad Z = \sum_{\sigma} \exp(-H(\sigma)).$$

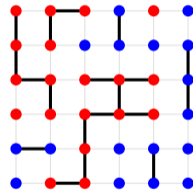
Ising model = 2-state Potts model

Fortuin–Kasteleyn q -random-cluster model

- configuration weight: $p^{\#\text{open}}(1-p)^{\#\text{closed}} q^{\#\text{clusters}}$.

- q vary continuously. $p_c = \frac{\bar{q}}{1 + \bar{q}}$.

- $q = 1$ case = Bernoulli bond percolation.



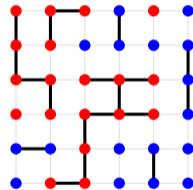
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Edwards–Sokal coupling: $p = 1 - e^{-\beta}$

- $\mathbb{P}_{\text{Potts}}[\sigma_x = \sigma_y] - q^{-1} = 1 - q^{-1} \mathbb{P}_{\text{random-cluster}}[x \text{ and } y \text{ are connected}]$.
- More generally, **q -state Potts spin-spin correlations are expressed by connectivity.**

Baxter Kelland Wu loop/height correspondence



From Duminil-Copin, Gagnebin, Harel, Manolescu, Tassion (2021)

Nienhuis (1984) for $q \in (0; 4]$; Duminil-Copin Kozłowski Lammers Manolescu (2026) for $q \in [1; 4]$

Height function converges to the Gaussian free field with variance $2 = \arccos^2(\frac{q}{4})$.

Nienhuis (1984)

Planarity: 2 points are connected \Leftrightarrow no loops in between.

change loop weight around a point to $2 \cos(\epsilon)$ \Leftrightarrow Add electric operator insertion $e^{i\epsilon GFF}$.

no-loop condition \Rightarrow loop weight $2 \cos(\epsilon) = 0 \Rightarrow \epsilon = \pi/2$.

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In Coulomb gas framework, e^{ieGFF} has scaling dimension

$$\frac{e^2 (1-g)^2}{2g}; \quad \text{with} \quad g = \frac{1}{\arccos(\frac{1}{q})};$$

One-arm exponent $\alpha_1 = (1 - 4(1-g)^2)/(8g)$.

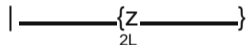
Bond percolation: $q = 1 \Rightarrow g = \frac{2}{3} \Rightarrow \alpha_1 = \frac{5}{48}$.

Ongoing work by Chen Duminil-Copin He Krachun Manolescu Xia for $q \geq 2$ [1; 4].

Transfer matrix T : keep track of loop configuration slice by slice in the q -random-cluster model.

Local join/split operations are generated by **Temperley Lieb algebra** with $n = \frac{q}{q-1}$:

$$E_m^2 = nE_m; \quad E_m E_{m+1} E_m = E_m:$$



$$T = \frac{q^{L-2}}{(1 + \frac{q}{q-1})^{2L-1}} \left(\prod_{m=1}^L (1 + E_{2m-1}) \right) \left(\prod_{m=1}^{L-1} (1 + E_{2m}) \right).$$

Partition function $Z = \text{Tr}(T^M)$:

Related to quantum group $U_q(\mathfrak{sl}_2)$; Yang Baxter.

From Jesper Lykke Jacobsen

Conjecture

Saleur Bauer (1989); Koo Saleur (1993); Folklore

Temperley Lieb algebra \Rightarrow Virasoro algebra.

Saleur Bauer (1989); Dubail Jacobsen Saleur (2010)

$$Z_{L;M}(n; n^0) = \prod_{j=0}^L \frac{\sin(2j+1)\theta}{\sin\theta} \text{tr}_{V_{2j}^L} T^M;$$

V_j^L : rep. space of length- $(2L)$ diagrams with j strings.

contractible loop weight: $n = e^{i\theta}$ \bar{q} . (percolation: $n = 1$.)

non-contractible loop weight: $q = 2 \cos \theta$.

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non-contractible loop weight: $\bar{q} = 2 \cos \cdot 0$.

Conjecture $\Rightarrow \lim \text{tr}_{V_{2j}^L} T^M = K_{1;1+j}$ as $L; M \rightarrow \infty$ and $L=M$.

$K_{1;1+j}(q)$: Virasoro character on Kac module with $q = \bar{e}$.

$$K_{1;1+j}(q) = q^{-\frac{c}{24}} \left(\prod_{k=1}^{\infty} (1 - q^k) \right)^{-1} \left(q^{\frac{gj^2}{4}} \frac{(1-g)^j}{2} - q^{\frac{g(j+2)^2}{4} + \frac{(1-g)(j+2)}{2}} \right).$$

Generating function for non-contractible loops

$\lim_{L, M \rightarrow \infty} Z_{L;M}(n; n^0) = Z(n; n^0)$ as $L, M \rightarrow \infty$ and $L=M \rightarrow \infty$.

$$Z_{q=1}(n; n^0) = \prod_{j=0}^{n-1} \frac{\sin(2j+1)\pi}{\sin \pi} K_{1;1+j}(q).$$

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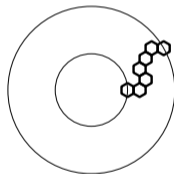
Annulus crossing formula

Cardy (2006); S. Xu Zhuang (2020)

$$P[\text{annulus crossing}] = \frac{Z_{q=1}(n; 0)}{Z_{q=1}(n; n)} = q^{\frac{-3}{2}} \frac{(6i) (3i)}{(2i) (3i)}$$

$$= \frac{1}{2} \log(R=r), \quad (z) = e^{i z=12} \prod_{n=1}^{\infty} (1 - e^{2ni z}).$$

1. recover $\frac{1}{2} = \frac{5}{48}$.
2. Cardy's derivation is based on Coulomb gas.



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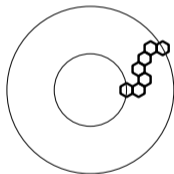
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Li Liu S. Zhuang (2026+) based on Ang Remy S. (2025)

We have $\lim_{L, M \rightarrow \infty} \text{tr}_{V_{2j}^L} T^M = K_{1;1+j}$ assuming scaling limits of bond percolation models on Z

Global strategy

Scaling limit of 2D critical percolation is encoded by SLE curves.

Exact solvability of SLE_{κ} \Rightarrow exact solvability of percolation.

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Loewner evolution

SLE = Loewner evolution with Brownian motion as the driving function.

Martingale observables, Itô calculus, 2nd order differential equation.

SLE coupled with Liouville quantum gravity

Mating of trees.

Exact solvability of Liouville conformal field theory.

Remark: so far almost orthogonal to the three CFT perspectives.

Scaling limit of interfaces in 2D critical lattice models should be SLE curves with $\kappa > 0$. Schramm (1999).

Scaling limit of the entire loop collection is described by Conformal Loop Ensemble (CLE)
She eld (2009), Werner She eld (2012).

Percolation corresponds to SLE_6 and CLE_6 .
Proved for site percolation on the triangular lattice.
Smirnov (2001), Camia Newman (2006).

SLE on H from 0 to 1

$$dg_t(z) = \frac{2}{g_t(z) W_t} dt:$$

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{|z|^2}\right) \text{ as } |z| \rightarrow \infty.$$

$$W_t = \rho B_t. \quad \mathbf{B_t: Brownian motion.}$$

g_t : Loewner map. W_t : driving function.

Recipe to exact solvability

Identify martingale observable \Rightarrow 2nd order differential equation (BPZ type).

SLE formulation of crossing event

: SLE₆ curve on H from x to 1.

hits $[1; 1)$ before $(1; 0]$

\emptyset crossing from $[0; x]$ to $[1; 1)$ occurs.

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Martingale observable

Z_t : cross ratio of $(0; t; 1; 1)$ as in $H \cap [0; t]$.

$f(x) := P_x$ hits $[1; 1)$ before $(1; 0]$:

$f(Z_t)$ is a martingale.

Itô's formula $\Rightarrow 3z(1-z)f''(z) + 2(1-2z)f'(z) = 0$.

$f(0) = 0$ and $f(1) = 1 \Rightarrow$ Cardy's formula.

Scaling exponent/fractal dimension

One-arm exponent $\alpha_1 = \frac{5}{48}$.
realized as the leading eigenvalue of
a BPZ type differential operator.

Polychromatic k-arm exponents:

bulk: $\frac{k^2 - 1}{12}$ boundary: $\frac{k(k+1)}{6}$.

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Connectivity probability on planar domain with boundary marked points

Lawler Schramm Werner, Bauer Bernard Kytölä, Dubédat, Zhan, Beliaev Viklund, Peltola Wu, ...

a theory for random surface with conformal geometry: $e^{\text{conformal factor}} (d^2x + d^2y)$.

conformal factor sampled from **Liouville conformal field theory** (central charge c)

LQG describes scaling limit of random triangulations under conformal embedding

Uniformly sampled triangulation $\Rightarrow c_L = 26$.

Coupled with conformal matter $\Rightarrow c_m + c_L = 26$.

c_m : matter central charge.

Uniform triangulation

Liouville conformal field theory

conformal embedding
 $\xrightarrow{\hspace{1.5cm}}$
 scaling limit

From Nicolas Curien

$$d_{\text{Euclidean}} = \text{KPZ}(d_{\text{quantum}}).$$

d : dimension of a planar fractal.

$$\text{KPZ}(x) = \frac{5}{3}x - \frac{1}{3}x^2 \text{ for pure gravity } (c = 26).$$

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KPZ derivation of $d_1 = \frac{5}{48}$

for a lattice region with n^2 vertices,
the size of boundary connecting cluster is $2n - 1$.

On random triangulation of same size, the answer is $7n - 4$.

$$d_{\text{Euclidean}} = 2 - d_1 = \text{KPZ}\left(\frac{7}{4}\right) = \frac{91}{48} \Rightarrow d_1 = \frac{5}{48}.$$

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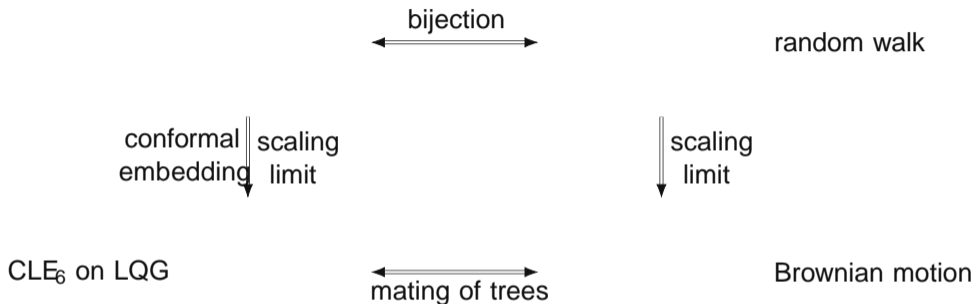
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Derived from the CFT description of LQG. (KPZ '88)

Justified by comparing to the discrete description.

Powerful framework to study random fractals. (Duplantier)



She eld (2010), Duplantier She eld (2011), Duplantier Miller She eld (2014), ...

Turn KPZ derivation of scaling exponents into rigorous proof.

Does not give correlation function/partition function information.

Overall strategy

Conformal factors of LQG surfaces are described by Liouville CFT.

Mating of trees gives **exact solvability of SLE/CLE coupled with Liouville CFT**.

Liouville CFT itself is exactly solvable.

Combining these two frameworks gives **solvability of SLE/CLE**.

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Exact solvability of Liouville CFT in probabilistic framework

DOZZ formula.

Kupiainen Rhodes Vargas (2017)

Conformal bootstrap.

Guillarmou KRV (2024, 2025)

Boundary counterpart.

Ang Remy S. Zhu (2023), Guillarmou Rhodes Wu (2026)

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Synergy between two frameworks was initiated by Ang, Holden, Remy, S. (2020 2022)

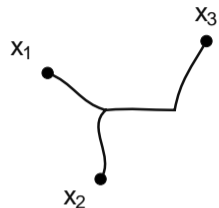
Recently applied to obtain exact solvability of percolation by S. et al. (2023-present)

$$G_n(x_1; x_2; \dots; x_n) := \lim_{\rho \rightarrow 0} \rho^{-n} P[x_1; x_2; \dots; x_n \text{ are connected}].$$

$$R = p \frac{G_3(x_1; x_2; x_3)}{G_2(x_1; x_2)G_2(x_1; x_3)G_2(x_2; x_3)} \quad 1 = \frac{5}{48}.$$

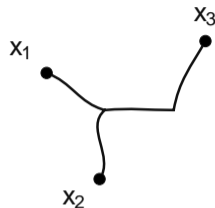
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Continuum limit of 3-point-connected percolation on random triangulation of the sphere

Approach 1: **CLE₆+LQG in the mating-of-trees framework; exactly solvable.**

Approach 2: **matter CFT Liouville CFT.**

(CFT description of LQG)

R is the matter 3-point correlation (properly normalized).

Liouville 3-point correlation = $\text{DOZZ}_{e_L} \left(\frac{1}{2}; \frac{1}{2}; \frac{1}{2} \right)$.

$q_L = 26$ and $\rho = \frac{5}{48} = \text{KPZ} \left(\frac{1}{2} \right)$.

Derivation of R

(Matter 3-point correlation Liouville 3-point correlation) is exactly solvable from mating-of-trees.

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Zamolodchikov (2005)

$\text{ImDOZZ}_{c_M}(\text{matter}_1; \text{matter}_2; \text{matter}_3) \text{DOZZ}_{c_L}(\text{Liou}_1; \text{Liou}_2; \text{Liou}_3) = 1.$ (up to normalization)

$c_M + c_L = 26$ and $\text{matter} = \text{KPZ}(\text{Liou})$.

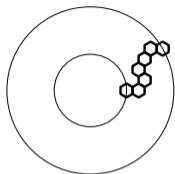
percolation: $c_M = 0$ and $c_L = 26$.

$$\text{matter}_1 = \text{matter}_2 = \text{matter}_3 = 1 = \frac{5}{48}.$$

$$\Rightarrow R = \text{ImDOZZ}_{c=0}(1; 1; 1).$$

Cardy (2006); S. Xu Zhuang (2024)

$$\begin{aligned}
 P[\text{annulus crossing}] &= \frac{q^{-\frac{3}{2}}}{2} \frac{(6i)(\frac{3}{2}i)}{(2i)(3i)} \\
 &= \frac{1}{2} \log(R=r), \quad (z) = e^{i z=12} \prod_{n=1}^{\infty} (1 - e^{2ni z}).
 \end{aligned}$$



Continuum limit of percolation with crossing on random triangulation of the annulus

Approach 1: CLE_6 + LQG in the mating-of-trees framework; exactly solvable.

Approach 2: **matter CFT** **Liouville CFT** **ghost CFT**.

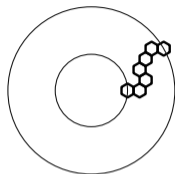
$$\int_0^{R_1} (Liouville theory on annulus of modulus) Z_{\text{matter}}() Z_{\text{ghost}}() d$$

$$Z_{\text{matter}}() = P[\text{annulus crossing}]$$

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Ang Remy S. (2025): **general method to derive random modulus** of a random annulus in LQG

Apply to random annulus with or without crossing event; compare to get $Z_{\text{matter}}()$.

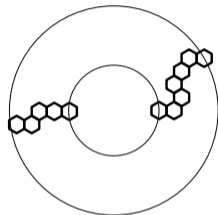
$P[\text{annulus crossing with two disjoint open arms}] \asymp r^{2+\theta(1)}$:

Nolin Qian S. Zhuang (2023)

θ_2 is the unique solution in the interval $(\frac{1}{4}, \frac{2}{3})$ to the equation

$$\frac{p \sqrt{36x+3}}{4} + \sin \frac{2}{3} \frac{p \sqrt{12x+1}}{3} = 0:$$

θ_2 is a transcendental number. $\theta_2 = 0.35666683671288$



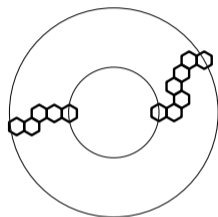
$P[\text{annulus crossing with two disjoint open arms}] \stackrel{?}{\sim} x^{2+\alpha(1)}$:

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No predictions previously other than a wrong guess $\frac{17}{48}$:

KPZ derivation for $\alpha_1 = \frac{5}{48}$ is hard to make work, (KPZ¹ (α_2) is transcendental).

Our proof relies on the disk analog of DOZZ formula for Liouville CFT.

Conformal bootstrap

spectrum resolution of Hamiltonian via Virasoro action
operator product expansion (OPE)

Development in physics

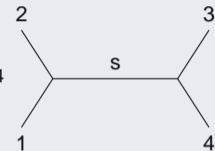
method: axiomatic bootstrap + transfer matrix.
exact predictions: 3-point function on sphere; 2-point function on disk.

Update: 1-point torus (Jesper Jacobsen's talk on Monday)

Development in mathematics

mating of trees + Liouville CFT solves 3-point functions beyond ImDOZZ .
method based on path integral for SLE reproduces 2-point disk prediction.

Conformal bootstrap expansion for 4-point sphere

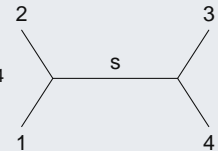
$$\langle V_1(z)V_2(0)V_3(1)V_4(1) \rangle = \sum_{s \in S} C_{12s} C_{s34} \mathcal{C}_{1234}^s$$


S: spectrum.

C_{ijk} : structure constant.

\mathcal{C}_{1234}^s : conformal block, determined by the Virasoro algebra.

Conformal bootstrap expansion for 4-point sphere

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The diagram shows a central horizontal line labeled 's'. From the left end of this line, two lines branch out to the left, labeled '1' (bottom) and '2' (top). From the right end of the line, two lines branch out to the right, labeled '4' (bottom) and '3' (top).

S: spectrum.

C_{ijk} : structure constant.

\mathcal{C}_{1234}^s : conformal block, determined by the Virasoro algebra.



The small diagram is identical in structure to the larger one in the equation above, showing a central channel 's' connecting two pairs of external legs (1,2 on the left and 4,3 on the right).

each expansion corresponds to a way of cutting a 4-punctured sphere into two 2-punctured disks
 crossing symmetry: consistency between different expansions.

Jacobsen, Nivesvivat, Ribault, Saleur et.al.

Assume percolation (more generally loop/cluster models) satisfy **conformal bootstrap**.

Annulus/torus partition functions **constrain the spectrum**.

Crossing symmetry **constrains structure constants**.

 further constraint by assuming degenerate elds.

 further constraint from Temperley Lieb algebra and variants.

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3-point function on sphere Jacobsen Nivesvlat Ribault-Roux (2021)

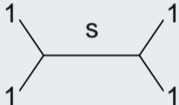
$$C_{123} = \frac{Y_{123}}{Y_{12} Y_{23} Y_{13}} + \frac{1}{2} \frac{X_{123}}{X_{12} X_{23}} + \frac{1}{2} \frac{X_{123}}{X_{13} X_{23}}$$

include imaginary DOZZ but encode **much more observables**.

e.g. **3 points on the same loop**.

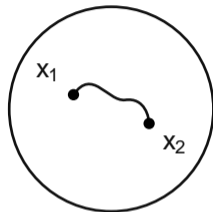
$$G_{\text{disk}}(x_1; x_2) := \lim_{l \rightarrow 0} l^{2k+1} P[x_1 \text{ and } x_2 \text{ are connected in disk}].$$

Downing Jacobsen Nivesvivat Ribault Saleur (2026)

$$G_{\text{disk}}(x_1; x_2) = \sum_{s \in S} C_{11s} R_s$$


$S = f(1; 2k+1) : k \in \mathbb{Z}_+ \text{ as in Kac table.}$

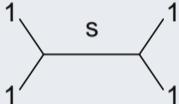
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$R_{(1; 2k+1)} = (1)^{k+1}$: disk 1-point function.

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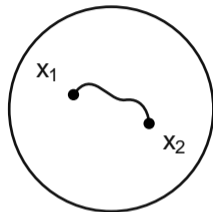
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Interaction with the probabilistic approach

This framework is **not fully analytic**, involves **guess-and-check** via numerical methods.

For the C_{123} formula, **probabilistic derivation** of the 3-point loop case played a crucial role in **navigating the guess-and-check progress**.

Path integral approach to SLE

View SLE as a measure of the form $e^{-\text{Action}[\gamma]}$ $D\gamma$.

conformal restriction.

Lawler Schramm Werner, Kontsevich Suhov, Dubédat, Zhan

large deviation, classical geometry.

Wang, Viklund Wang, Takhtajan Teo.

integration by parts \Rightarrow Virasoro action.

Baverez Jegou; Gordina Qian Wang

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A probabilistic argument for 2-point disk correlations

Jin S. Wu (2026+)

based on the path integral approach to SLE.

not rigorous, but there is no guess-and-check.

rigorous path integral so far only for SLE with $\kappa \in (0; 4]$.

we adopt it to percolation and analytically continue to SLE

Can solve 2-point disk functions that have not been predicted in physics.

Significant progress in physics and mathematics suggest that 2D critical percolation is indeed governed by an exactly solvable CFT.

Complete understanding has not been achieved. (n-point sphere is not pinned down)

The next step is to **combine all methods to reach a physically complete picture.**

- Significant progress in physics and mathematics suggest that 2D critical percolation is indeed governed by an exactly solvable CFT.
- Complete understanding has not been achieved. (n -point sphere is not pinned down.)
- The next step is to **combine all methods to reach a physically complete picture.**

Rigorous understanding of CFT perspectives remains a major mathematical challenge.

- connection to **minimal model**?
- rigorous **Coulomb gas** and connection to **imaginary Liouville theory**?
- connection to **Yang–Baxter/quantum group** integrability from **Temperley–Lieb algebra**?
- rigorous **path integral** framework for SLE_6 and CLE_6 ?
- deeper understanding of $c < 1$ **conformal block and log CFT**?
- Where does **backbone exponent** fit in? monochromatic k -arm with $k = 3$?

