

Near-Critical Dimer Model: Massive SLE and massive field theory



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Criticality and conformal invariance

- Consider a lattice model of **statistical mechanics**, say Ising.
- At “critical point”, emergence of **conformal symmetries**
Belavin-Polyakov-Zamolodchikov (1984)
→ **Conformal Field Theory**
- Away from critical point, exponential decay of correlations, **trivial scaling limits**.

Criticality and CFT

- However, **close to** the critical point:
 - non-trivial scaling limits;
 - conformal on infinitesimally small scales,
 - but exponential decay of correlations on large scales.
- No conformal invariance → general **Quantum Field Theories**.
- Often, a model can be viewed either *geometrically* (random curve describing interfaces) or through a *random field* (height function).
 - These two complementary points of view lead to very different descriptions. We will try to examine both here.

Criticality and CFT

- Makarov-Smirnov (2010):

The key property of SLE is its conformal invariance, which is expected in 2D lattice models only at criticality, and the question naturally arises: Can SLE success be replicated for off-critical models?

In most off-critical cases to obtain a non-trivial scaling limit one has to adjust some parameter [...], sending it at an appropriate speed to the critical value. Such limits lead to massive field theories....,

Off-criticality

- Geometric perspective: Makarov and Smirnov developed a family of examples called **massive SLE** (no general theory yet, as far as I'm aware).
Also gave partial arguments for convergence of some discrete models to massive SLE.
- Field theoretic perspective: **Sine-Gordon model**.
Remarkably, a QFT that is *not* conformal yet still **integrable**.
- We will illustrate these ideas with recent results **near-critical dimer** model:

Near-critical dimer model

- **Dimer model** (near critical point) \leftrightarrow **massive SLE₂**
 \leftrightarrow **sine-Gordon** model
- Predicted by **Lukyanov** (1997) in the context of the 6-vertex model.
(Precise formulation for dimers: B.-Haunschmid Sibitz).
- Joint works with **Levi Haunschmid Sibitz**, and **Scott Mason, Lucas Rey** respectively.



Levi Haunschmid Sibitz

Scott Mason



Lucas Rey



Overview

- I: *Review*

Massive LERW, and massive SLE.

Brief review of some examples

Dimer model, Temperleyan setup.

Review of earlier results at criticality

- II: *Results*

Near-critical model

Statement of results (informal)

Sine Gordon model, Coleman transform.

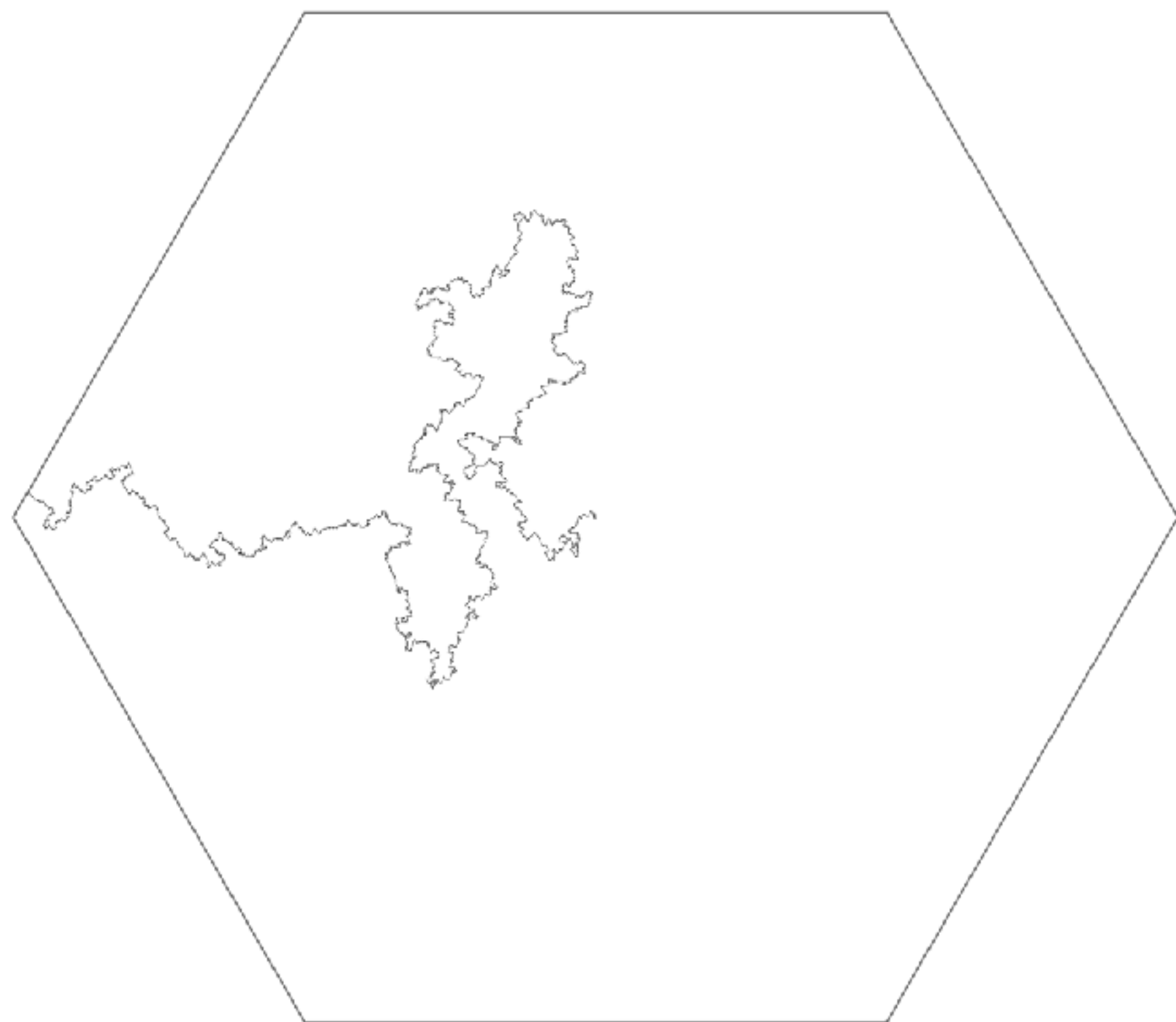
Notion of massive holomorphicity

I. Background. Massive LERW

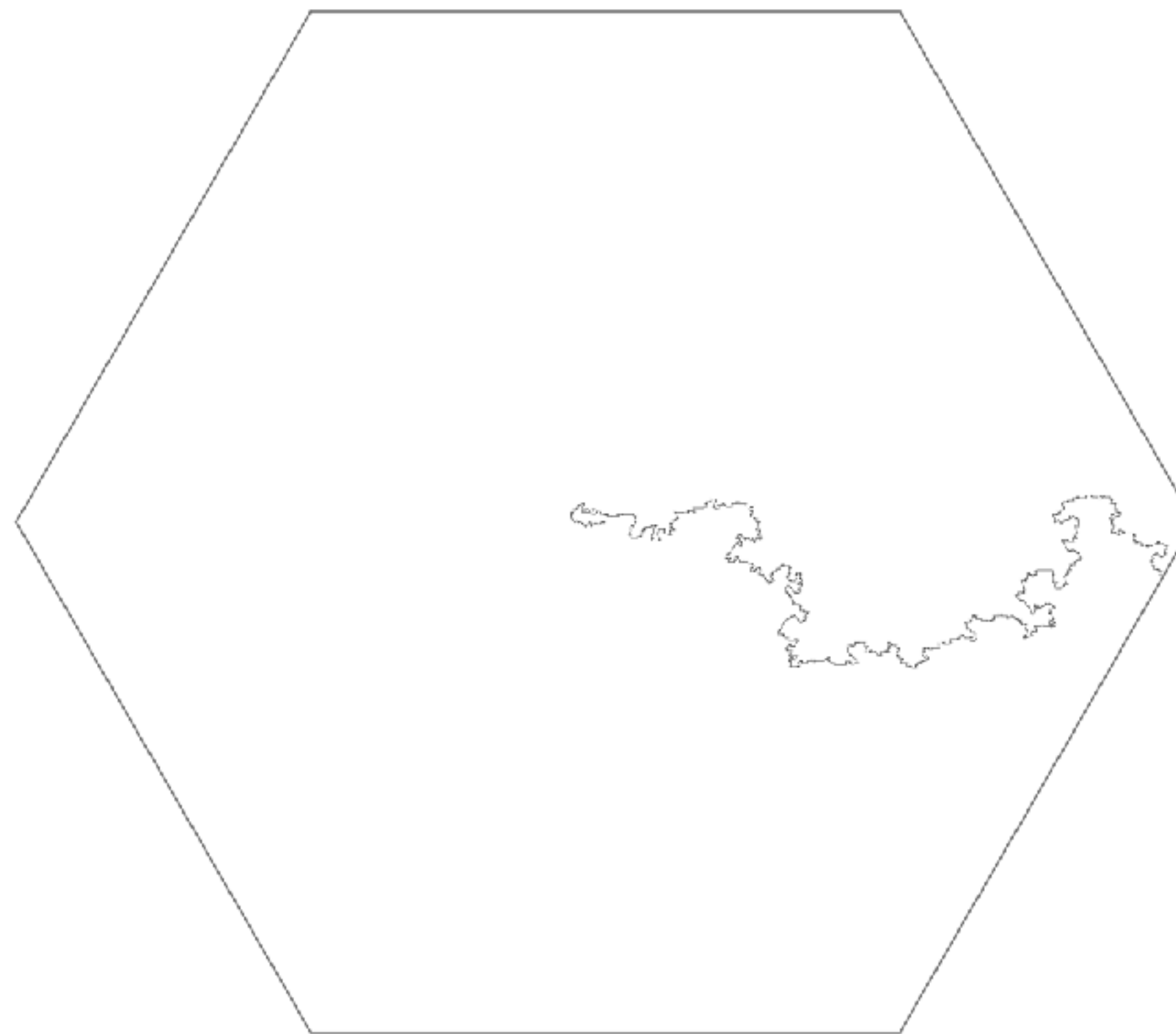
Makarov-Smirnov, Bauer-Bernard-Kytolla

- Consider $(X_n^\epsilon, n = 0, 1, \dots, \tau_\Omega) = \text{RW}$ in Ω_ϵ , starting from $z_0 \in \Omega$, killed outside Ω .
- As the mesh size $\epsilon \rightarrow 0$, $(X_{\text{speed-up}(t)}^\epsilon)_{t \geq 0} \rightarrow \text{BM}$.
- **Loop-erasure**: $\text{LE}(X^\epsilon)$ converges to a random simple curve, (radial) SLE_2 (**Lawler-Schramm-Werner**, 2002).
- Now consider an **off-critical perturbation**: let $m \in \mathbb{R} = \text{“mass”}$. Suppose at each step, $X^{m,\epsilon}$ is killed with proba $m^2\epsilon^2/2$, else continues to random neighbour.
- Condition $X^{m,\epsilon}$ to leave Ω before being killed. What is scaling limit of its LERW?

Massive LERW



An ordinary LERW



A massive LERW

Example: massive LERW

Makarov-Smirnov, Bauer-Bernard-Kytolla

- **Theorem** (Makarov-Smirnov '08, Bauer-Bernard-Kytolla '07, Chelkak-Wan '14)

$\text{LE}(X^{m,\epsilon}) \rightarrow \text{mSLE}_2$ or **massive SLE** as $\epsilon \rightarrow 0$.

- (**Note:** absolutely continuous wrt SLE_2 . This is expected for all massive scaling limits corresponding to SLE_κ with $\kappa \leq 4$.)

- Given $\kappa > 0$, one gets martingale observables of SLE_κ of the form

$$M_t(z) = Z_t(z)^\beta Z_t'(z)^\sigma,$$

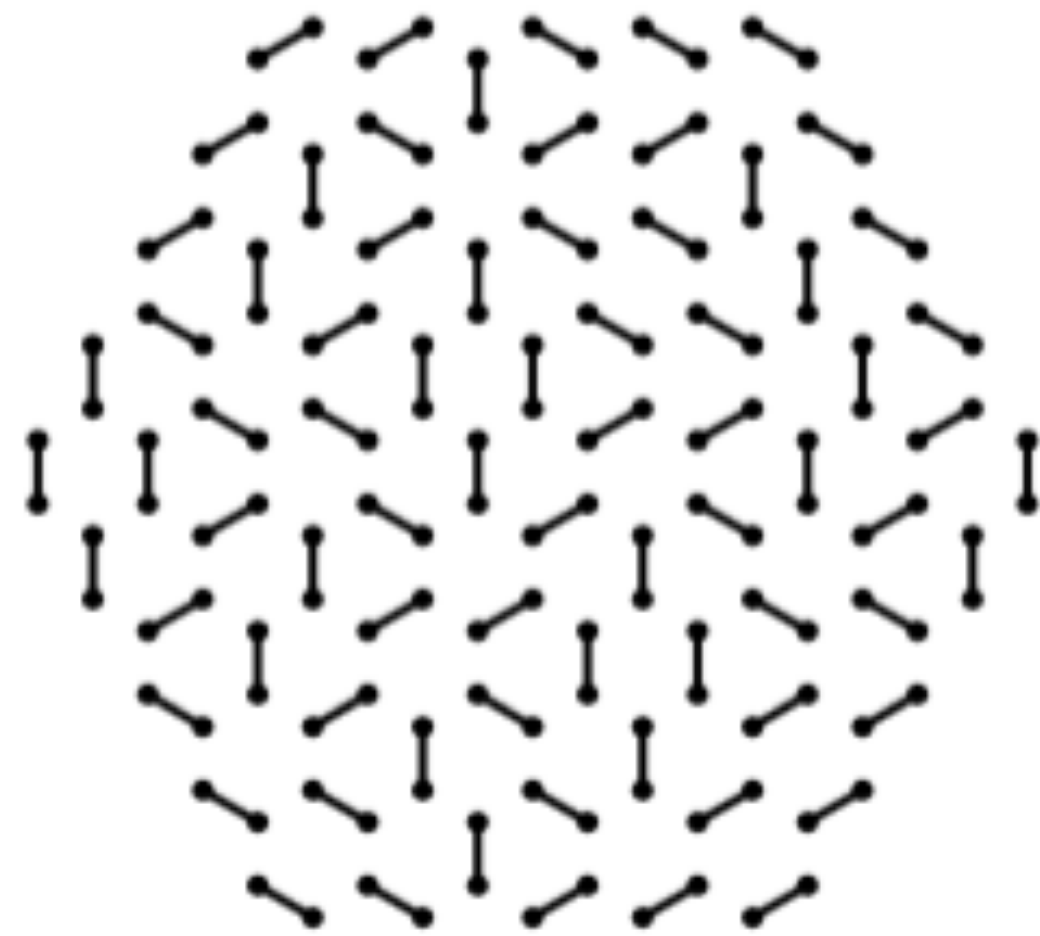
where $Z_t(z) = g_t(z) - \xi_t$ is the centered Loewner flow, and $\sigma = \beta + \frac{\beta(\beta - 1)}{4}\kappa$

- There is a natural way to associate to each such choice of M a “massive” analogue; massive SLE is “defined” to be a curve for which this observable is a martingale.

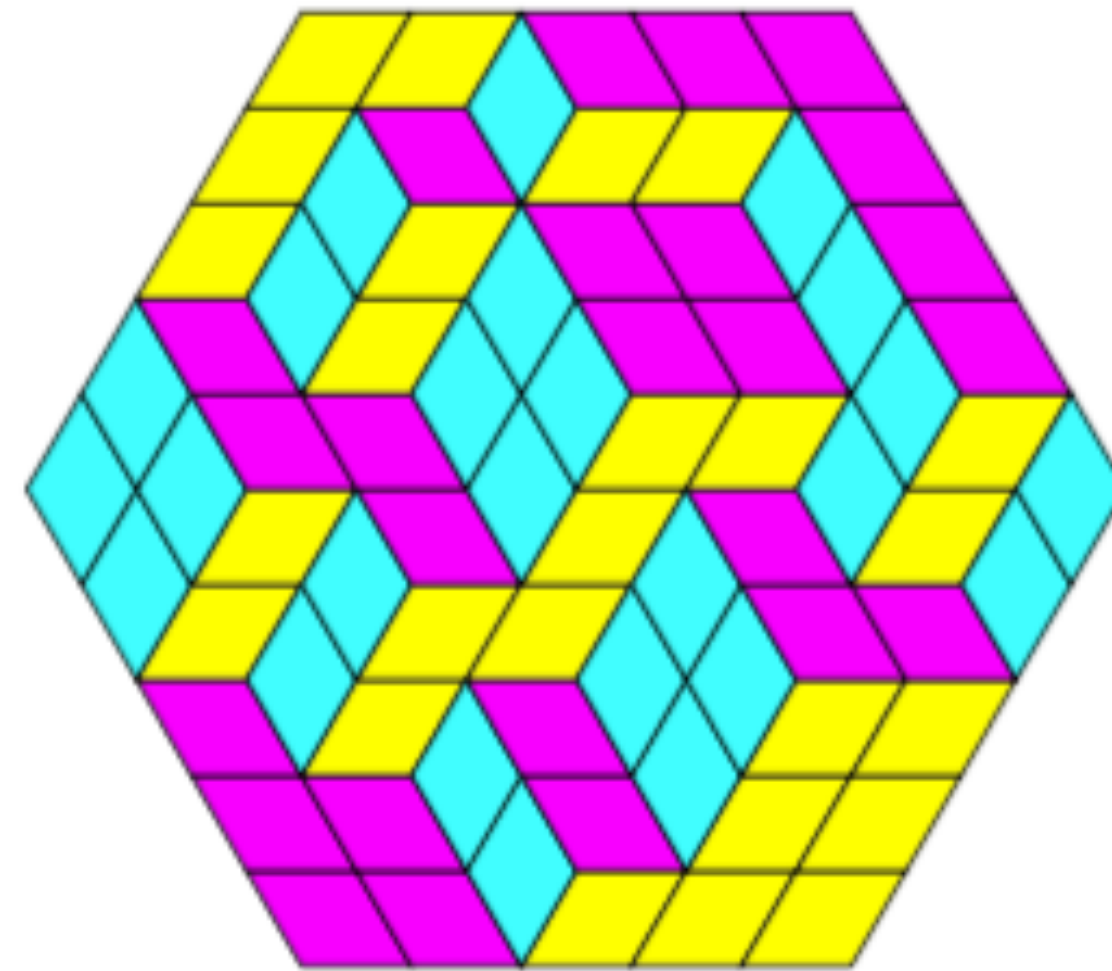
Other examples

- Harmonic explorer \rightarrow $mSLE_4$ (Makarov-Smirnov '10, **Papon '23**)
- Level lines of **massive GFF** are $mSLE_4$, also coincide with **massive Brownian loop soup cluster outer boundary** (**Papon '23**)
- Interfaces of magnetically perturbed Ising model, a form of $mSLE_3$ (**Papon '24**)

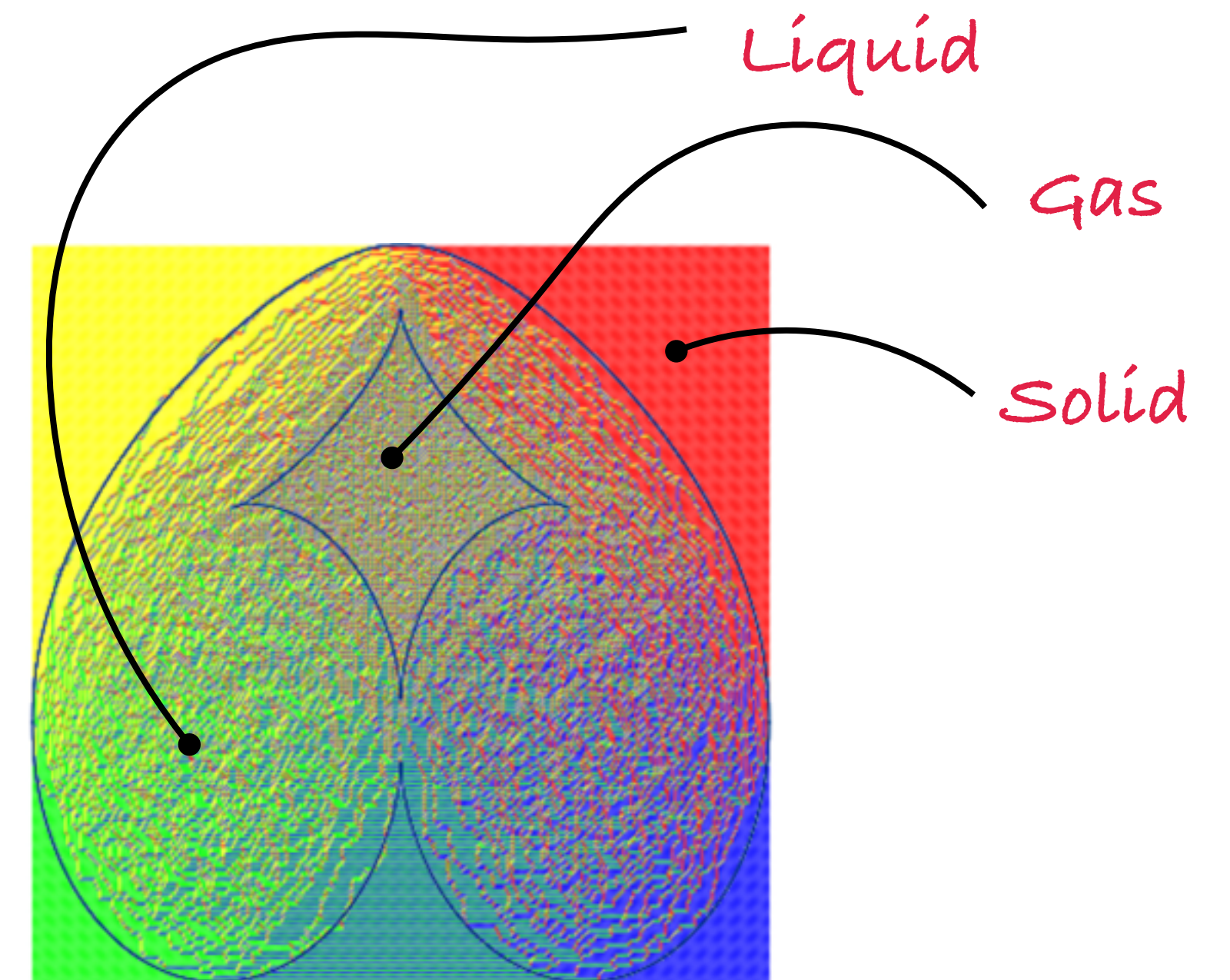
I. Background on dimers



Perfect Matching
(aka dimer model)



Associated (rhombus) tiling

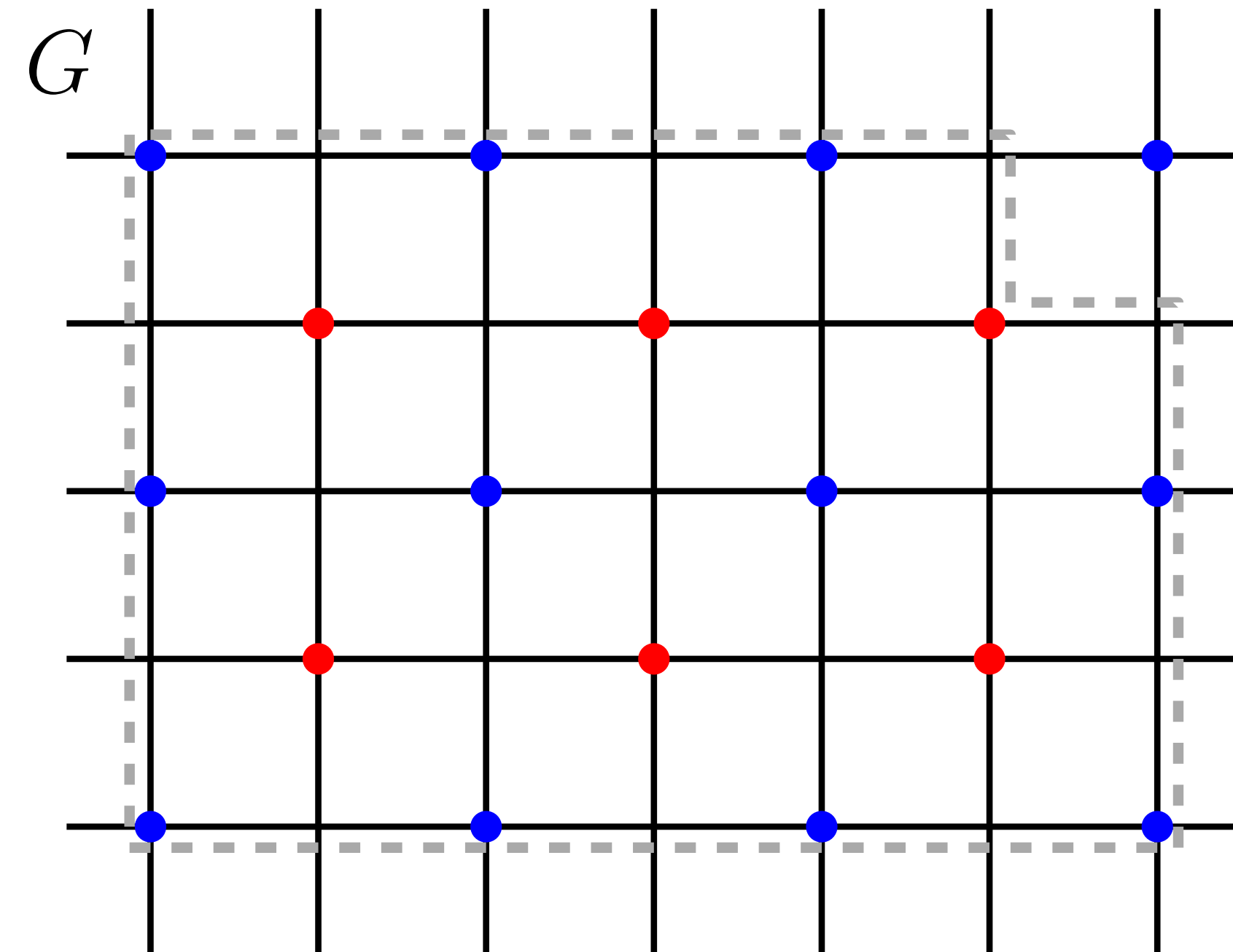


Multiple phases:
Kenyon, Okunkov, Sheffield

Temperleyan domains

- Square lattice \mathbb{Z}^2 . Black-white coloring.
- Black vertices \rightarrow \bullet , \bullet
- **Temperleyan boundary conditions:**
 \rightarrow on the boundary only black vertices of type 1 (say **blue**) vertices are exposed, one corner removed.
- Given edge weights $(c_e)_{e \in E}$ on G , consider **dimer model**:

$$\mathbb{P}(\mathbf{m}) = \frac{1}{Z} \prod_{e \in \mathbf{m}} c_e.$$



Conformal Invariance

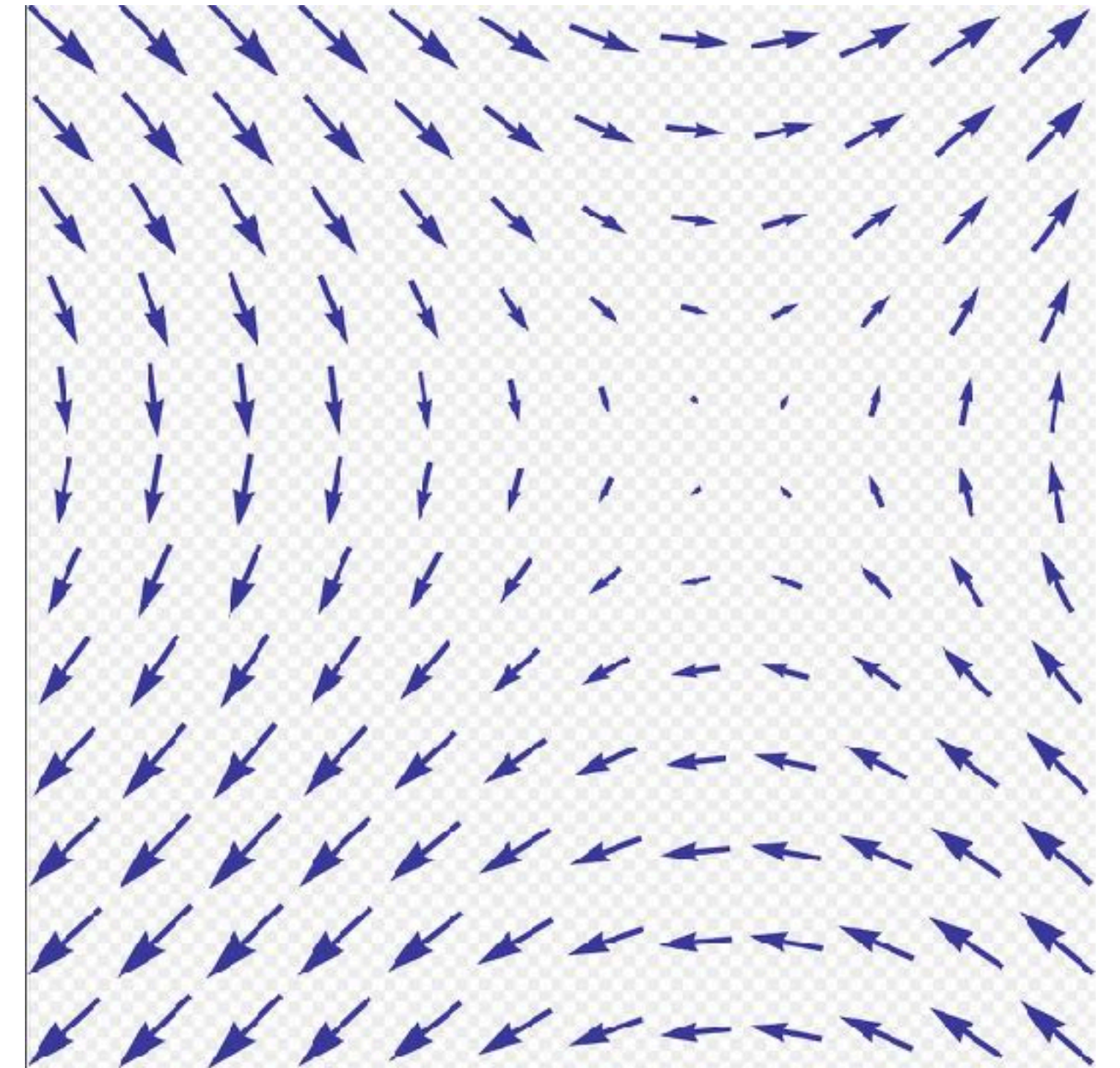
- **Kenyon** (1999): on square lattice with **unit weights**, dimer model is asymptotically conformally invariant.

More precisely: associated **centered height function** converges to $(1/\sqrt{\pi}) \times$ **GFF** with Dirichlet boundary conditions.

- First proved on square lattice, then **de Tilière** (2007) full plane isoradial superpositions, **Li** (2014), assuming straight edge portion. See also **B.-Laslier-Ray** (2020).

II. New results for dimers

- Near-critical dimer model introduced by **Chhita** (2012)
(square lattice, whole plane, homogeneous: $\alpha = \text{const.}$).
Non-Gaussianity.
- Basic parameter for deformation: $\alpha : \Omega \rightarrow \mathbb{R}^2 \simeq \mathbb{C}$ a
vector field: **external electromagnetic field**.
- We always assume $\alpha = \nabla V$ derives from a **potential**.



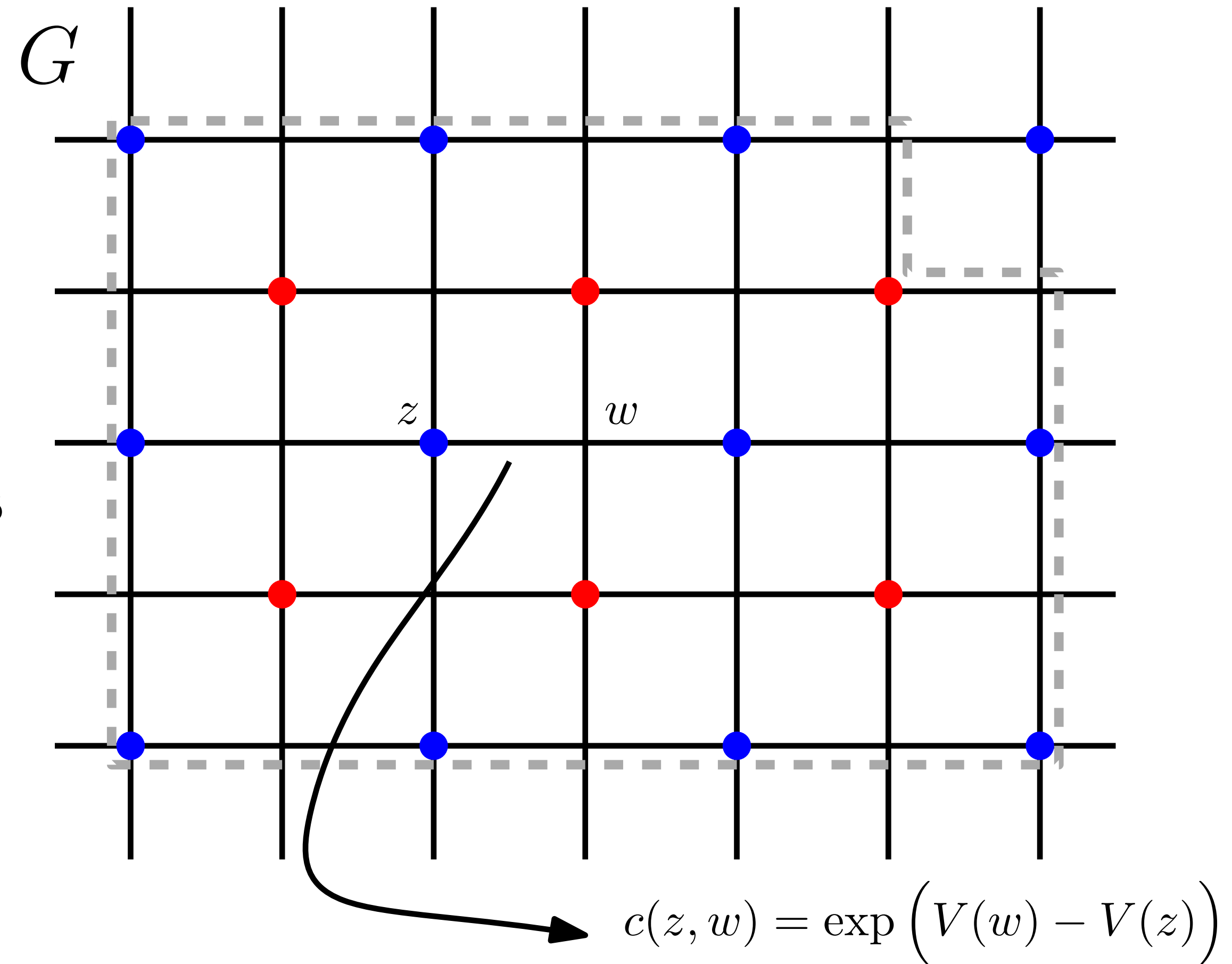
Near Critical Dimer model

- **Near-critical weights:**
Around each **black** vertex $z \in B$,
take weights $c(z, w) = \exp(V(w) - V(z))$
- These are unoriented, but can also be viewed as oriented weights on Γ .

Lemma: As $\epsilon = \text{mesh size} \rightarrow 0$,

SRW on Γ converges to **Langevin diffusion:**

$$dX_t = dB_t + \nabla V(X_t)dt$$



Near Critical Dimer model

- **Near-critical weights:**

Around each **black** vertex $z \in B$,

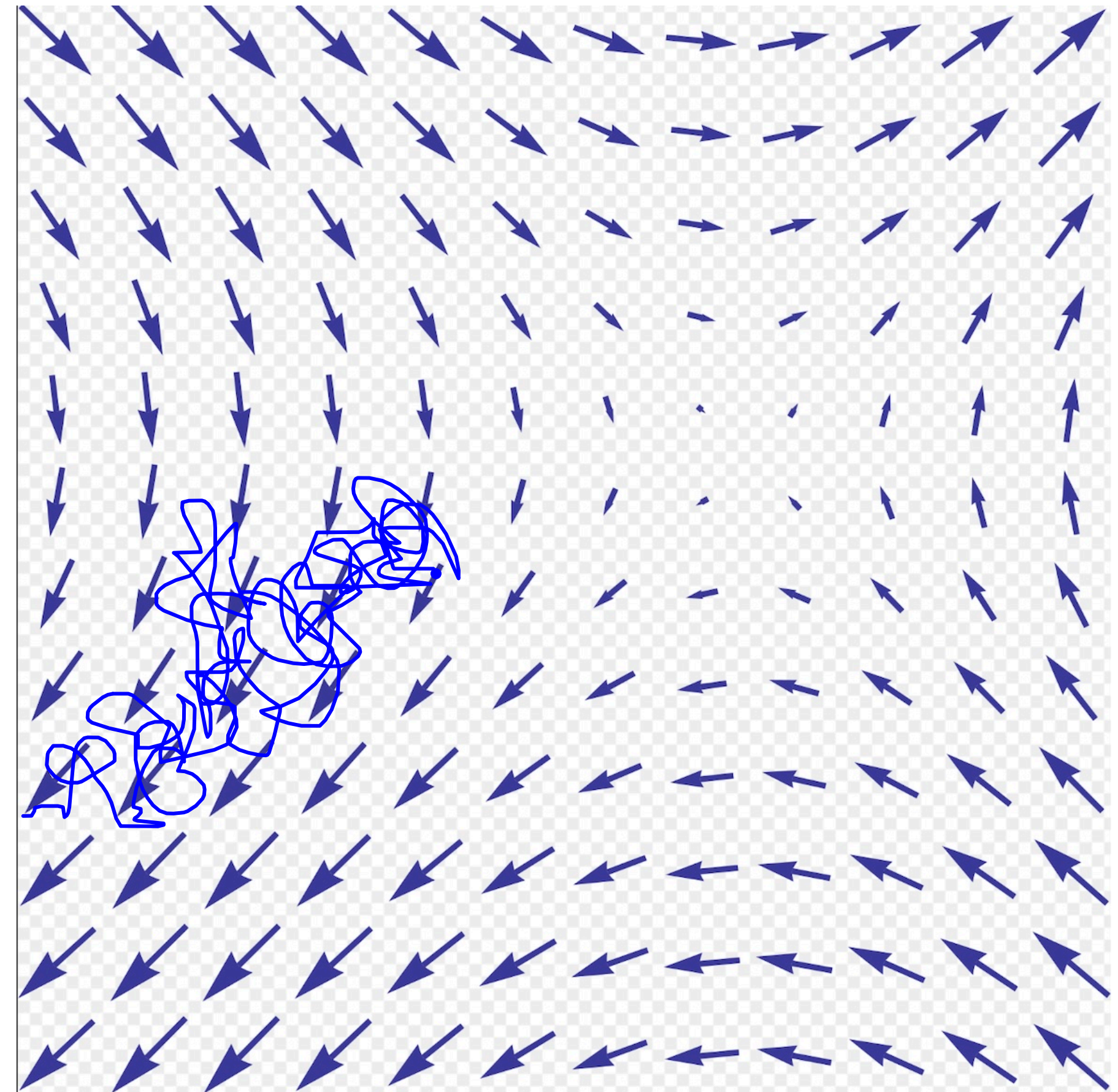
take weights $c(z, w) = \exp(V(w) - V(z))$

- **Lemma:** As $\epsilon = \text{mesh size} \rightarrow 0$,

SRW on Γ with these directed weights

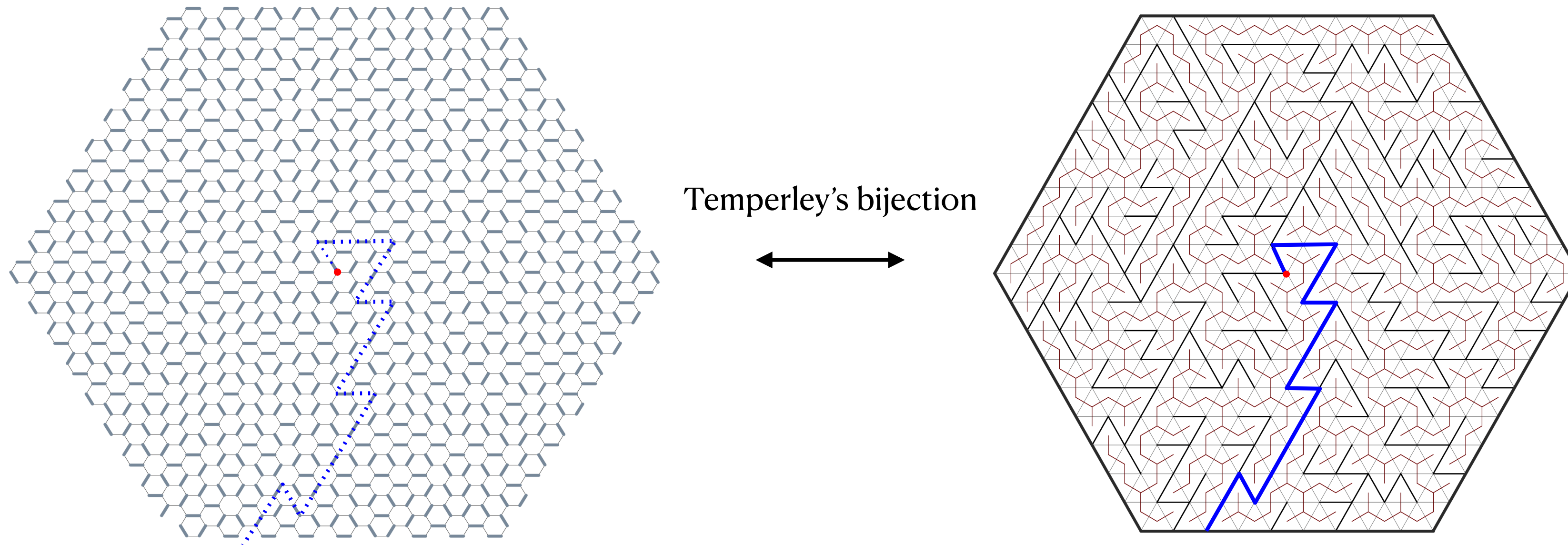
converges to **Langevin diffusion:**

$$dX_t = dB_t + \nabla V(X_t)dt$$



Connection to Massive SLE

- **B.-Haunschmid Sibitz ('22):** Let $\alpha = \nabla V$, and V **log-convex**: $\mu := \Delta V + \|\nabla V\|^2 \geq 0$ in Ω
Temperleyan paths converge to a form of **massive SLE₂**.
($\alpha = \text{const.}$: Makarov-Smirnov, Chelkak-Wan. Otherwise, new).



- Existence and universality of scaling limit for height function
- This limit height function satisfies **conformal covariance**. Conjectured precise link to SG.

Connection to Sine-Gordon

- We again assume that $\alpha = \nabla V$, and V is **log-convex**: $\mu := \Delta V + \|\nabla V\|^2 \geq 0$ in Ω . This function plays the role of the “**mass**”. (This is conformally invariant.)
- **Theorem 1 (B.-Mason-Rey '26)** Assume that $\partial\Omega$ contains a straight edge portion, or the image of such a domain by a map which is analytic in a neighbourhood of Ω .

Let $h_\epsilon(z)$ denote the centered height function associated to near-critical dimers.

Then $h_\epsilon \rightarrow \phi$ in law, in the sense that $(h_\epsilon, f) \rightarrow (\phi, f)$ in distribution for arbitrary test functions.

Furthermore...

Weak sense

Main result

- Furthermore, for every $(s_1, \dots, s_n) \in \{0, 1\}^n$, every $z_1, \dots, z_n \in \Omega$ pairwise \neq ,
 $\mathbb{E}[\partial^{(s_1)}\phi(z_1)\dots\partial^{(s_n)}\phi(z_n)] = \det \left(F_{s_i, s_j}(z_i, z_j) \right)_{1 \leq i \neq j \leq n}$ where:
- $\partial^{(s_i)}$ stands for **Wirtinger derivatives**, with $\partial^{(0)} = \partial$, $\partial^{(1)} = \bar{\partial}$.
- $F_{s_i, s_j}(w, z)$ are **massive holomorphic** functions of $w \in \Omega$ (see below) and satisfy a (squared) Riemann boundary condition.

- That is, $\mathbb{E}[\phi(\zeta_1)\dots\phi(\zeta_n)] = \sum_{s \in \{0, 1\}^n} \int_{\gamma_1} \dots \int_{\gamma_n} \det \left(F_{s_i, s_j}(z_i, z_j) \right)_{1 \leq i \neq j \leq n} \prod_{i=1}^n dz_i^{(s_i)}$

Connection to sine-Gordon

- **Theorem 2 (B.-Mason-Rey '26).** Now suppose $\Omega = \mathbb{D}$ and α satisfying additional technical assumptions.

$$\mathbb{E}[\partial^{(s_1)}\phi(z_1)\dots\partial^{(s_n)}\phi(z_n)] = \lambda^n \langle \prod_{i=1}^n \partial^{(s_i)}\phi(z_j) \rangle_{SG(\rho)}$$

Still need to define precisely

with $\lambda = - (4\sqrt{\pi})^{-1}$, $\rho = - \frac{\tilde{\alpha}}{8\mathbf{i}\pi}$ (which is purely real by assumption),

$\tilde{\alpha}(z) = e^{\mathbf{i} \arg g'(z)} \alpha(z)$, where $g : \mathbb{H} \rightarrow \mathbb{D}$.

Moment Problem

- The previous theorems identify the moments of $\mathbb{E}[(\phi, f)^n]$.
- If X is a RV then $\log \mathbb{E}(e^{i\mu X}) = \sum_n (i\mu)^n \kappa_n / n!$, where $\kappa_n = \kappa_n(X) = n$ th cumulant
- **Theorem 3 (B.-Mason-Rey'26)**: the above series has positive radius of convergence.
- This gives a **Fredholm regularised determinant** (of order 4) representation for the log-mgf of the sine-Gordon field.

Sine Gordon model

- Informally : $\mathbb{P}_{\text{SG}(\rho)}(d\phi) = \exp\left(c_0 \int_{\Omega} \cos(\sqrt{\beta}\phi(z))\rho(z)dz\right) \mathbb{P}^{\text{GFF}\#}(d\phi),$

$\rho \in C^\infty(\Omega, \mathbb{R})$ was related to α in the theorem.

Normalisation conventions are important,

$$\mathbb{E}^{\text{GFF}\#}(\phi(x)\phi(y)) = \frac{1}{2\pi} \log |x - y|^{-1} + O(1)$$

Sine Gordon model

- Assigning rigorous meaning to $\mathbb{P}_{\text{SG}(\rho)}$ is far from straightforward.
- As $\beta \nearrow$, model goes through **sequence of thresholds** of increasing difficulty.
- First threshold: $\beta = 4\pi$, known as **Free Fermion point**. Thresholds accumulate at $\beta = 8\pi$.
- For $\beta < 4\pi$, one can directly make sense of
$$\int_{\Omega} \cos(\sqrt{\beta}\phi(x))\rho(x)dx := \lim_{\epsilon \rightarrow 0} \epsilon^{-\beta/4\pi} \text{Re} \left(\int_{\Omega} e^{i\sqrt{\beta}\phi_{\epsilon}(x)}\rho(x)dx \right)$$

(\rightarrow **Imaginary Multiplicative Chaos**, cf. Junnilla—Saksman—Webb).
- Then $\mathbb{P}_{\text{SG}(\rho)}$ is well defined and is $\ll \mathbb{P}^{\text{GFF}\#}(\text{d}\phi)$.

Free Fermion point of SG

- At $\beta = 4\pi$, $\int_{\Omega} \cos(\sqrt{\beta}\phi(x))dx$ ceases to make sense as function of ϕ
- Nevertheless, still possible to make sense of \mathbb{P}^{SG} (but presumably NOT $\ll \mathbb{P}^{\text{GFF}\#}(d\phi)$)
- At FF point, SG has additional symmetries: **Coleman's correspondence.**
- **Sine - Gordon (ρ) \leftrightarrow massive Free Fermions**

$(\psi(z))_{z \in \Omega} = (\psi_1(z), \psi_2(z))_{z \in \Omega} =$ **Dirac fermions** on Ω (Grassmann variables)

with formal density $\exp\left[-\int_{\Omega} dx (\bar{\psi} \not{\partial}\psi + \rho\bar{\psi}_1\psi_1 + \bar{\rho}\bar{\psi}_2\psi_2)\right] = \exp(-S(\bar{\psi}, \psi))$, where

$S(\bar{\psi}, \psi) = \int_{\Omega} \bar{\psi}(z)(\not{\partial} + \alpha)\psi(z)dz$, and where $\not{\partial} =$ **Dirac operator**: $= \begin{pmatrix} 0 & 2\bar{\partial} \\ 2\partial & 0 \end{pmatrix}$.

Free Fermion point of SG

- This is a **massive extension** of the **boson-fermion** correspondence:
if one takes $\rho = 0$ then $\text{GFF} \leftrightarrow \exp \left[- \int_{\Omega} dx \bar{\psi}(x) \not{\partial} \psi(x) \right]$

- Note: SG becomes “**trivial**” on the fermionic side!

$$\exp \left[- \int_{\Omega} dx \left(\bar{\psi} \not{\partial} \psi + \rho \bar{\psi}_1 \psi_1 + \bar{\rho} \bar{\psi}_2 \psi_2 \right) \right] = \exp(-S(\bar{\psi}, \psi))$$

- Constructing the RHS rigorously is relatively standard in mathematical physics: eg Benfatto, Falco, and Mastropietro (2007).

This is a particular case of the **Thirring model** which in general even allows for a quartic interaction.

Coleman correspondence

- Let $(\psi, \psi^*) =$ associated **Majorana fermions**:

$$\psi = \psi_1 + \mathbf{i}\psi_2; \psi^* = \psi_1 - \mathbf{i}\psi_2$$

$$\mathbb{E}_{\text{SG}(\rho)} \left(\prod_{i=1}^n \partial^{(s_i)} \phi(z_i) \right) = \left\langle \prod_{i=1}^n \psi^{s_i}(z_i) \right\rangle, \text{ (up to factors of } \mathbf{i} \text{ and } \sqrt{\pi}), \text{ with } \psi^0 = \psi, \psi^1 = \psi^*$$

- and $\langle \cdot \rangle =$ “expectation” under this measure: eg,

$$\begin{aligned} \langle \psi^{s_i}(w), \psi^{s_j}(z) \rangle &= \int \psi^{s_i}(w) \psi^{s_j}(z) e^{-S(\bar{\psi}, \psi)} d\bar{\psi} d\psi \\ &= (\not{\partial} + \rho)_{s_i, s_j}^{-1}(w, z) \end{aligned}$$

- (this is a concrete quantity, **except for boundary conditions** which remain unspecified!)

Weak sense

Berezin / Grassmann integration

Fermionic Wick rule

- As a consequence of Coleman's correspondence, it's possible to evaluate the n -point function of Sine-Gordon in terms of its two-point functions:

- This gives:

$$\mathbb{E}_{\text{SG}(\rho)} \left(\prod_{i=1}^n \partial^{(s_i)} \phi(z_i) \right) = \det \left(\langle \psi^{s_i}(z_i), \psi^{s_j}(z_j) \rangle \right)_{1 \leq i \neq j \leq n}$$

- This Fermionic Wick rule is convenient for the identification of SG in the scaling limit, since we get determinants for the limit of the Wirtinger derivatives of the height function.

SG - what is rigorous?

- In the **unit disc** $\Omega = \mathbb{D}$, and with $\rho \in C_c^\infty(\Omega)$,
Park-Virtanen-Webb (2025) following **Bauerschmidt-Webb (2020)** in the plane
construct sine-Gordon model at FF point rigorously and show Coleman relation:
 - Starting from a white noise regularisation of the GFF ϕ_ϵ , let
 - : $\cos(\sqrt{\beta}\phi_\epsilon) := c_\rho \epsilon^{-\beta/(4\pi)} \cos(\sqrt{\beta}\phi_\epsilon)$ denote Wick-renormalised (c_α explicit but not universal)
 - :: $\cos(\sqrt{\beta}\phi_\epsilon) ::= \cos(\sqrt{\beta}\phi_\epsilon) : - A_\epsilon$ where A_ϵ is explicit *and* universal
- and let $\langle F \rangle_{\text{SG}(\epsilon|\rho)} \propto \mathbb{E}_{\text{GFF}\#} \left[F(\phi) \exp \left(\int_{\Omega} :: \cos(\sqrt{\beta}\phi_\epsilon(x)) :: \rho(x) dx \right) \right]$

SG - what is rigorous?

- **Theorem (Park-Vertanen-Webb, 2025)**

$$\lim_{\epsilon \rightarrow 0} \langle \prod_{i=1}^n \partial^{(s_j)} \phi(z_j) \rangle_{\text{SG}(\epsilon|\rho)} \text{ exists, and } = \det \left(\langle \psi^{s_i}(z_i), \psi^{s_j}(z_j) \rangle \right)_{1 \leq i \neq j \leq n}$$

where ψ obeys the boundary conditions

$$\overline{\langle \psi(w) \psi(z) \rangle} = \tau(w) \langle \psi(w) \psi^*(z) \rangle; w \in \partial \mathbb{D}$$

- This constructs $\text{SG}(\rho)$ in $\Omega = \mathbb{D}$ with $\beta = 4\pi$ (free fermion point) and shows Coleman transform with $\rho \in C_c^\infty(\mathbb{D}; \mathbb{R})$.

Massive Holomorphicity

- Identify $\alpha : \Omega \rightarrow \mathbb{C}$ to complex valued function, we define **massive holomorphicity** as follows:

$$\bar{\partial}f = \frac{1}{2}\alpha\bar{f}$$

Generalizes notion introduced by Makarov-Smirnov and studied by Park ($\alpha = \text{const.} \in i\mathbb{R}$).

- **Lemma:** if f is an α -massive holomorphic function, then $\Re(f)$ is massive harmonic, i.e., $(\Delta - \mu I)\Re(f) = 0$, where $\mu = \Delta V + \|\nabla V\|^2 \geq 0$ is our mass function.
- We do not know if a massive harmonic function always has a conjugate. Fortunately, in the case of the inverse Kasteleyn matrix below, this is true.

Kasteleyn theory

- By definition, discrete derivatives of $h_e \rightarrow$ occupancy of edges by dimers.
- To understand correlation of height derivatives \rightarrow dimer-dimer correlations.
- By Kasteleyn theory, $\mathbb{P}(e_1, \dots, e_n \in \mathbf{m}) = \det \left(K^{-1}(b_i, w_j) \right)_{1 \leq i, j \leq n}$,

where $K(w, b) =$ **Kasteleyn matrix** = (signed) adjacency matrix of G , and $\left(K^{-1}(b, w) \right)_{b \in B, w \in W}$ its inverse.

- **Key:** show convergence of $K^{-1}(b, w)$ in the scaling limit to a pair of complex functions, depending on the combinatorial type of the black vertex b .
Intuitively this gives rise to the two fermions ψ_1, ψ_2 at a point.

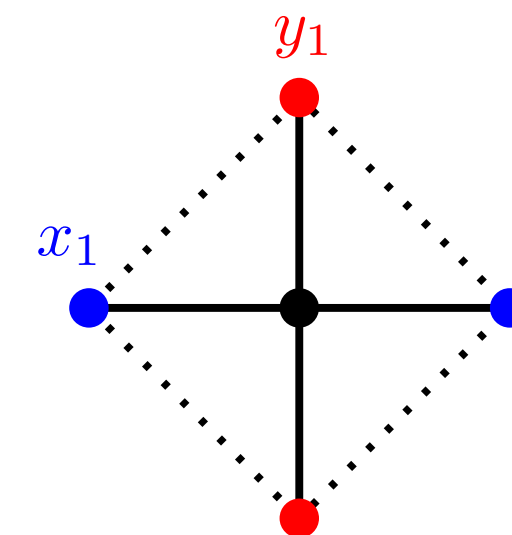
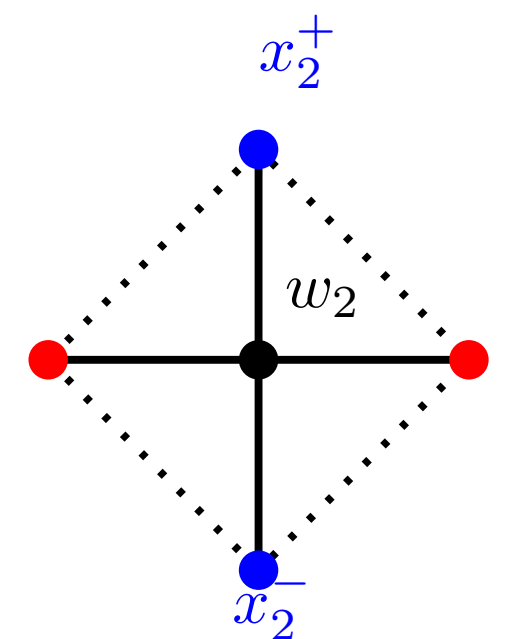
Scaling limit for K^{-1}

- **Theorem.** Assume $\partial\Omega$ has straight edge portion. Let $\epsilon =$ mesh size.
- \exists pair of (complex) functions (κ, κ^*) , massive holomorphic conjugates,
- $\kappa(w_1, w_2) =$ (massive) derivative of G^m

Fix $x_1 \in B$, $y_1 \in B$, and $w_2 \in W$. Then, uniformly away from diag:

$$K^{-1}(x_1, w_2) = \langle \bar{\kappa}(x_1, w_2), x_2^+ - x_2^- \rangle + o(\epsilon)$$

$$K^{-1}(y_1, w_2) = i \langle \bar{\kappa}^*(y_1, w_2), x_2^+ - x_2^- \rangle + o(\epsilon).$$



Scaling limit idea

- It is not hard to see that when $x \in B$, then $K^{-1}(x, w)$ is the discrete derivative of the massive Green function G^μ .
- Using tools from discrete complex analysis (extended to the massive setting) this gives a scaling limit for K^{-1} in this situation.
- More complicated if $y \in B$. If $\alpha = 0$, then $KK^{-1} = I$ so K^{-1} is discrete holomorphic, so derivatives in y -direction are $i \times$ derivatives in x -direction. Integrating these derivatives together with earlier asymptotics gives the conclusion.
- In the massive setting, this is not so simple.

Scaling limit idea

- In continuum, $f = u + iv$ is α -massive holomorphic iff **Cauchy-Riemann** equations:

$$e^V \nabla (ue^{-V}) = e^{-V} (-i) \nabla (ve^V)$$

ie, gradients of u and v are related by rotation and conjugation by $e^{\pm V}$

- We find an exact, discrete form of this identity which is satisfied by K^{-1}

Thanks!

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Discrete random structures: enumeration and scaling limits