RANDOM HYPERBOLIC SURFACES

a biased survey

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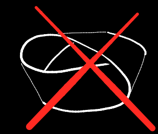
Université PARIS-SACLAY + erc

HYPERBOLIC SURFACE : DEFINITIONS



METRIC : 2d. Riemannian manifold cst curvature $\equiv -1$.

MODEL : 2d. manifold with charts $\to \mathbb{H}^2$

ALGEBRA : Quotient of $\mathbb{H}^2$ by subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$

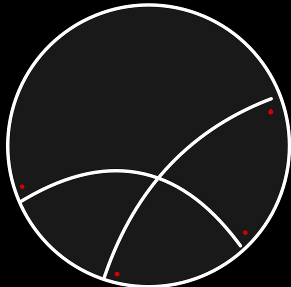
UNIFORMIZATION : Riemann surface with cover $\mathbb{D}$.

HANDS-ON :
$$\left.\begin{array}{l}\text{Gluing of pairs of pants.}\\ \text{Gluing of triangles.}\end{array}\right\} \text{“combinatorial”.}$$

...

GLUING PAIRS of PANTS

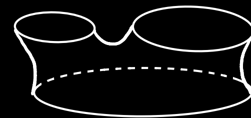
$$\frac{4\,|dz|^2}{(1-|z|^2)^2}$$



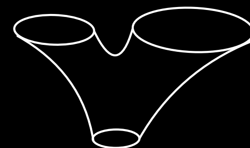
$\mathbb{D}$

HYPERBOLIC PLANE $\mathbb{H}^2$

b $\qquad$ c



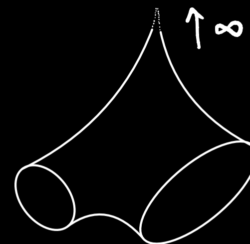
a



"PUNCTURE"

0

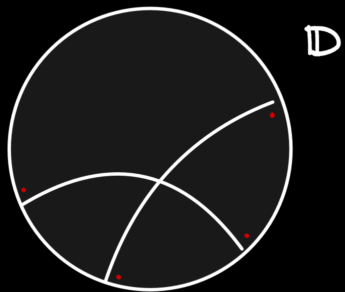
$\uparrow \infty$



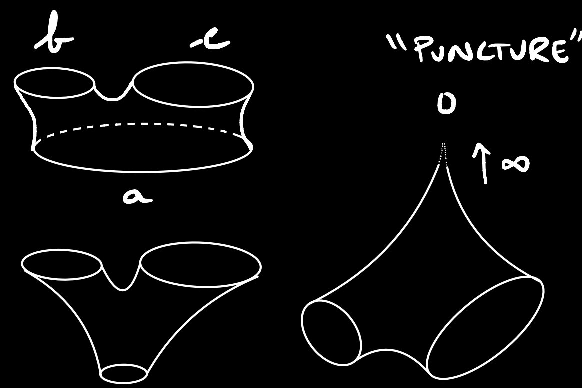
$a, b, c \in \mathbb{R}_+$

GLUING PAIRS of PANTS

$$\frac{4\,|dz|^2}{(1-|z|^2)^2}$$



HYPERBOLIC PLANE $\mathbb{H}^2$



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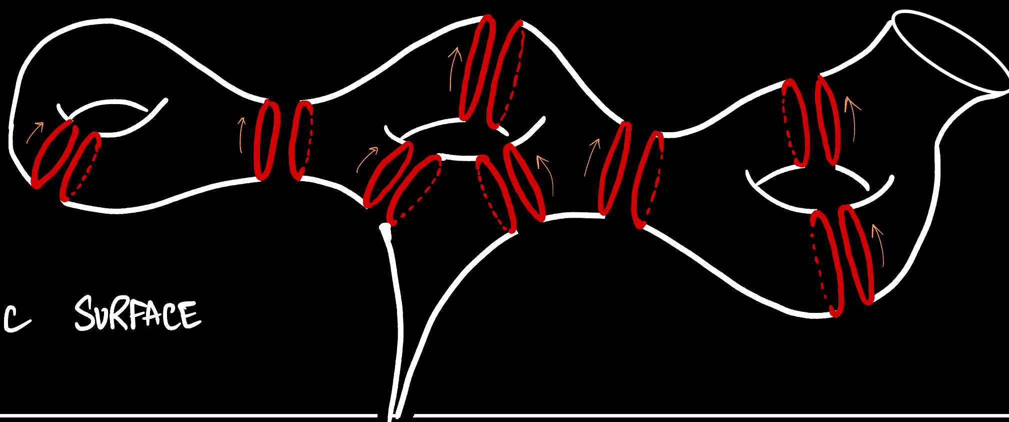
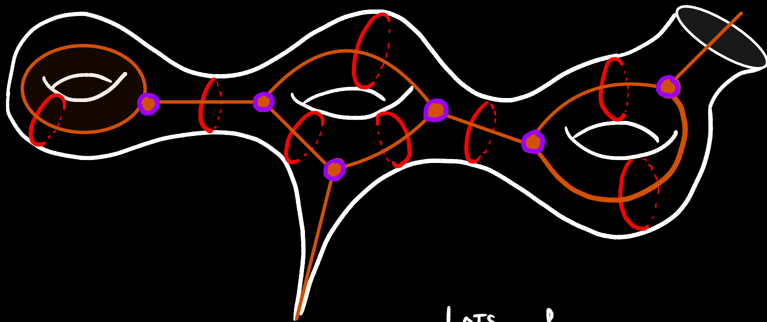
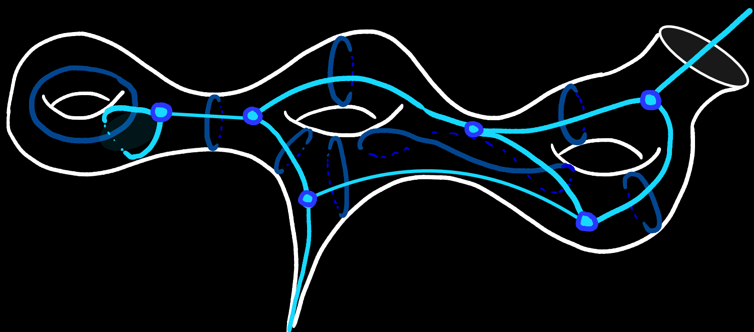


HYPERBOLIC SURFACE

MODULI SPACE



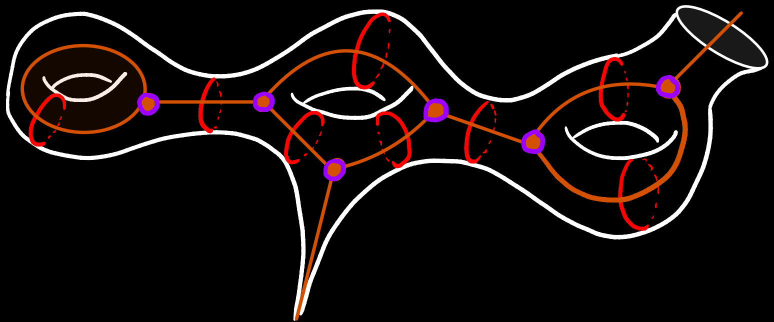
$=$ LOTS of IDENTIFICATIONS



MODULI SPACE

$$\mathcal{M}_{g,\ell_1,\ell_2\cdots\ell_n} = \left\{ \begin{array}{l} \text{genus } g \text{ hyp surface} \\ \text{with geo boundaries} \\ \text{length } \ell_i \quad i \in [\![1,n]\!] \end{array} \right\}$$

$\ell_i = 0$ PUNCTURES



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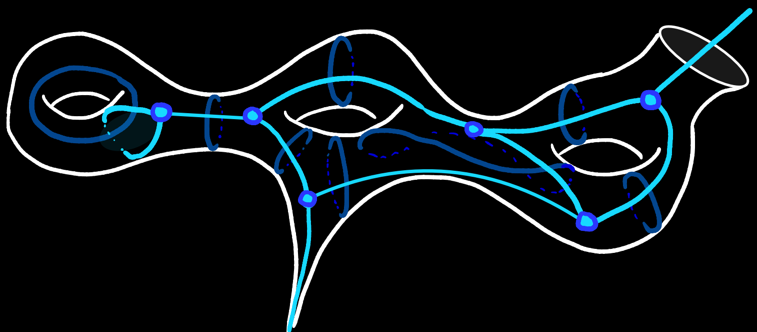

MODULI SPACE

$$\mathcal{M}_{g,l_1,l_2\dots l_n}$$

$= \left\{ \begin{array}{l} \text{genus } g \text{ hyp surface} \\ \text{with geo boundaries} \\ \text{length } l_i \quad i\in[\![1,n]\!] \end{array} \right\}$

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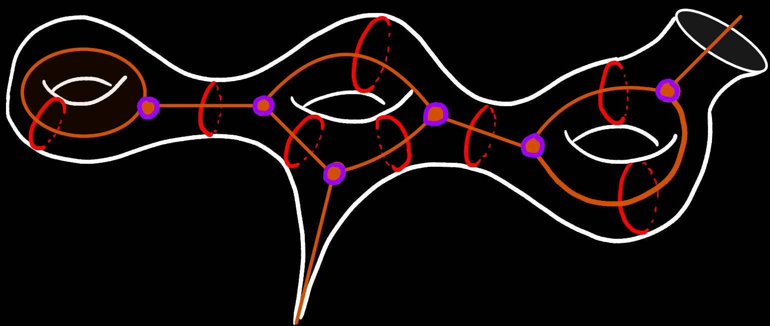
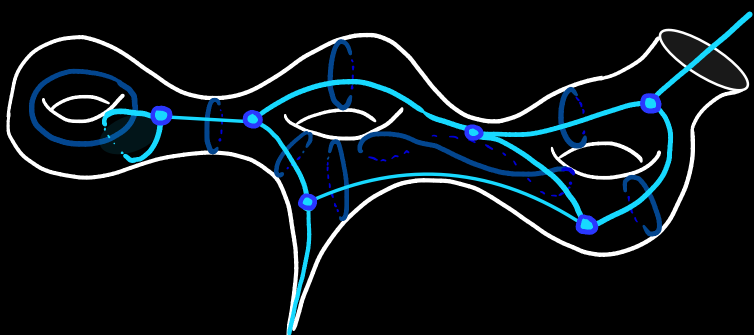


$=$

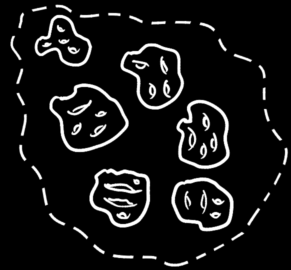


$$\boxed{\begin{array}{ll} 6g-6+2n & \text{PARAMETERS} \\ 2g+n-2 & \text{P.of.P.} \end{array}}$$

$\mathcal{M}_g \equiv \mathcal{M}_{g,0} \quad g \geq 2$

$\mathcal{M}_{0,n} \equiv \mathcal{M}_{0,\overbrace{000\dots0}^{n}}$

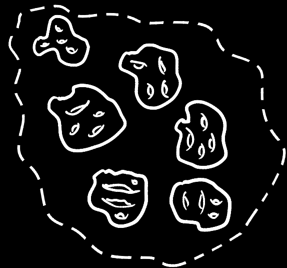
TAKE HOME MESSAGE

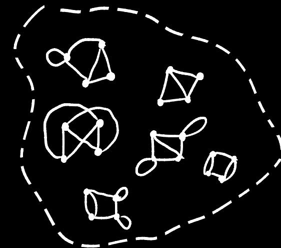


$\mathcal{M}_g \simeq$ COMPACT MANIFOLD

DIMENSION $6g-6$

TAKE HOME MESSAGE

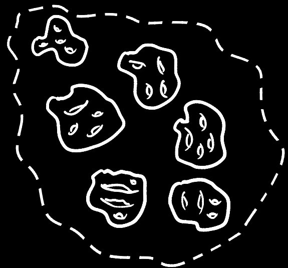


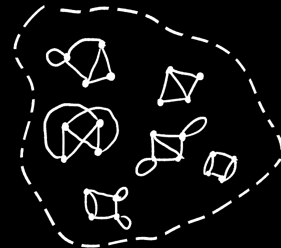


$\mathcal{M}_g \simeq$ COMPACT MANIFOLD DIMENSION $6g-6$ $\approx$ SPACE OF TRIVALENT MAPS $2g-2$ NODES

BOLLOBAS ERDÖS ALON
LUBOTSKY - PHILLIPS - SARNAK
FRIEDMAN HUANG - MCKENZIE - YAU

TAKE HOME MESSAGE





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$\mathcal{M}_{0,n} \simeq$ COMPACT MANIFOLD DIMENSION $2n-6$ $\simeq$ SPACE OF TRIVALENT PLANAR MAPS $n-2$ NODES

TUTTE, SCHAEFFER, LE GALL, MIERMONT
MILLER - SHEFFIELD ...

FIX A GEOMETRIC QUANTITY ϕ, WHAT IS

EXTREMAL PROPERTIES

$$\inf_{S_g \in \mathcal{M}_g} \phi(S_g)$$

$$\sup_{S_g \in \mathcal{M}_g} \phi(S_g)$$

OR

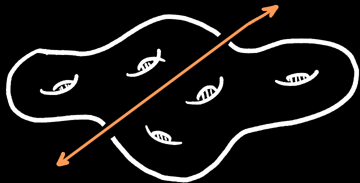
TYPICAL PROPERTIES

BEHAVIOR OF $\phi(S_g)$

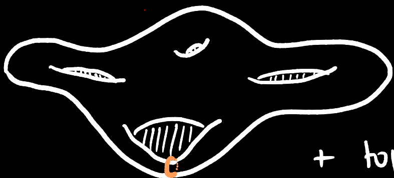
WHERE

$S_g \in \mathcal{M}_g$ RANDOM

GEOMETRIC OBSERVABLES

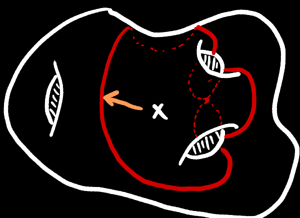


DIAMETER



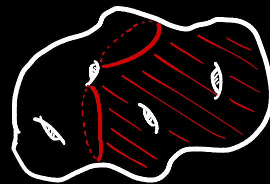
SYSTOLE

+ top. variants



INJECTIVITY RADIUS.

LENGTH SPECTRUM

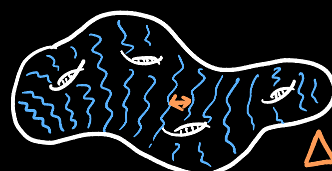


CHEEGER CONSTANT

$$\inf_{|A|<\frac{1}{2}|S|} \frac{|\partial A|}{|A|}$$

TOTAL PANTS LENGTH

min



SPECTRAL GAP

Δ - SPECTRUM

$\lambda_1(\Delta)$

A FEW RECENT DETERMINISTIC RESULTS

	inf on $\mathcal{M}_g$	sup on $\mathcal{M}_g$
DIAMETER	$\gtrsim \log g$	∞
SYSTOLE	0	$\lesssim 2 \log g$
CHEEGER CST	0	> 0 EXPANDERS ≤ 1
SPECTRAL GAP	0	> 0 EXPANDERS $\lesssim 1/4$
TOTAL PANTS LENGTH	0	$\gtrsim g \log g$ $\lesssim g^2$

A few recent deterministic results

	inf on $\mathcal{M}_g$	sup on $\mathcal{M}_g$
Diameter	$\gtrsim \log g$	∞
Systole	0	$\leq 2 \log g$ $\geq \frac{4}{3} \log g$ Buser Sarnak '94
Cheeger cst	0	> 0 $\quad \leq 1$
Spectral gap	0	> 0 $\quad \leq \frac{1}{4}$
Total pants length	0	$\gtrsim g \log g$ $\quad \lesssim g^2$

A Few Recent Deterministic Results

	inf on $\mathcal{M}_g$	sup on $\mathcal{M}_g$
Diameter	$\gtrsim \log g$	∞
Systole	0	$\leq 2 \log g$ $\gtrsim \frac{4}{3} \log g$ Buser Sarnak 94'
Cheeger cst	0	> 0 ≤ 1
Spectral gap	0	> 0 $\leq 1/4$
Total pants length	0	$\gtrsim g \log g$ $\lesssim g^2$ $\gtrsim g^{7/6-\varepsilon}$ Guth Parlier Young 10'

A Few Recent Deterministic Results

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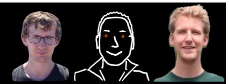
Diameter	$\gtrsim \log g$  $\log g$ 2019	∞
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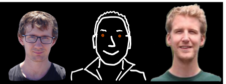
Diameter	$\gtrsim \log g$  $\log g$ 2019	∞
Systole	0	$\leq 2 \log g$ $\geq 4/3 \log g$ Buser Sarnak 94'
Cheeger cst	0	> 0 ≤ 1
Spectral gap	0	> 0 Expanders $\leq 1/4$ $1/4$ Hide Magee 23'
Total pants length	0	$\gtrsim g \log g$ $\lesssim g^2$ $\gtrsim g^{7/6-\varepsilon}$ 10' Guth Parlier Young

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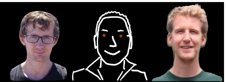
Diameter	$\gtrsim \log g$  $\log g$ 2019	∞
Systole	0	$\lesssim 2 \log g$ $\gtrsim 4/3 \log g$ Buser Sarnak 94'
Cheeger cst	0	> 0 $\leq 2/\pi$ ≤ 1 25' 
Spectral gap	0	> 0 Expanders $\leq 1/4$ $1/4$ Hide Magee 23'
Total pants length	0	$\gtrsim g \log g$ $\lesssim g^2$ $\gtrsim g^{7/6-\varepsilon}$ 10' Guth Parlier Young

MEASURES ON $\mathcal{M}_g$

DISCRETE "COMBINATORIAL" CONSTRUCTIONS 

- RANDOM PANTS DECOMPOSITION
- GLUING ALONG RANDOM TRIANGULATION (BROOKS-MAKOVER)
- RANDOM COVERS ...

USEFUL TO PROVE EXTREMAL PPTY, ERDÖS

MEASURES ON $\mathcal{M}_g$

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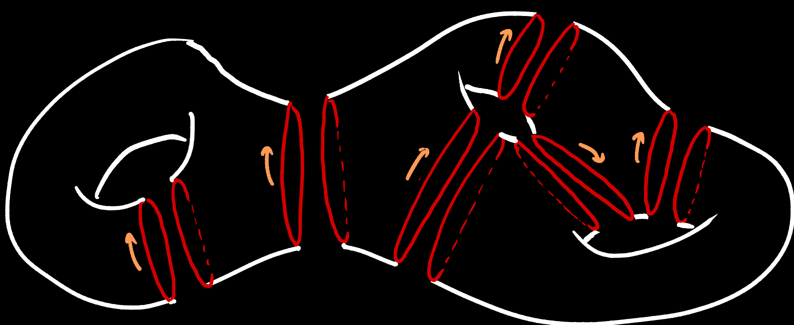
WEIL-PETERSSON MEASURE : "LEBESGUE MEASURE" ON $\mathcal{M}_g$.

FENCHEL-NIELSEN COORDINATES (WOLPERT 80').



FIX PANTS DECOMPOSITION

$$\mathbb{R}^{6g-6} \longrightarrow \mathcal{M}_g$$

$$\text{Leb} \longrightarrow \text{WP}$$

FINITE MEASURE.

RANDOM WEIL-PETERSSON HYPERBOLIC SURFACES, HIGH GENUS

MIRZAKHANI

RECURSIONS FOR $V_{g,n} = WP(\mathcal{M}_{g,n})$

$$V_{g,n_0} \underset{g \to +\infty}{\sim} \mathrm{Cste}\,(2g-3+n)!\,(4\pi^2)^{2g-3+n}$$

MIRZAKHANI
ZOGRAF.

07', 09', 10', 13'
...

RANDOM WEIL-PETERSSON HYPERBOLIC SURFACES, HIGH GENUS

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MIRZAKHANI ZOGRAF.

$\longrightarrow$ New proof of WITTEN'S CONJECTURE

$\longrightarrow$ RANDOM WP SURFACES in $\mathcal{M}_g$ HAVE:

- CHEEGER $(S_g) > 0{,}099$
- SPEC GAP $(S_g) > 0{,}02$
- DIAMETER $(S_g) < 40 \log g$.
- INJECTIVITY RAD $(S_g) > 1/6 \log g$.

WHP.

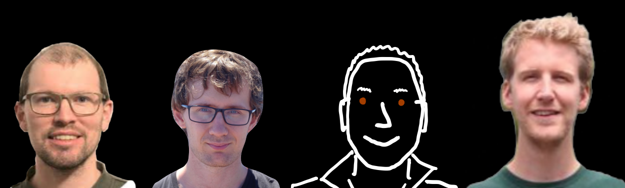
07', 09', 10', 13' ... ...

RANDOM WEIL-PETERSSON HYPERBOLIC SURFACES, HIGH GENUS

REINTERPRETING MIRZAKHANI / MCSHANE

→ PEELING HYPERBOLIC SURFACES

BUDD, BUDZINSKI, C., PETRI 2026+



RANDOM WEIL-PETERSSON HYPERBOLIC SURFACES, HIGH GENUS

REINTERPRETING MIRZAKHANI / MCSHANE

→ $\boxed{\text{PEELING}}$ HYPERBOLIC SURFACES

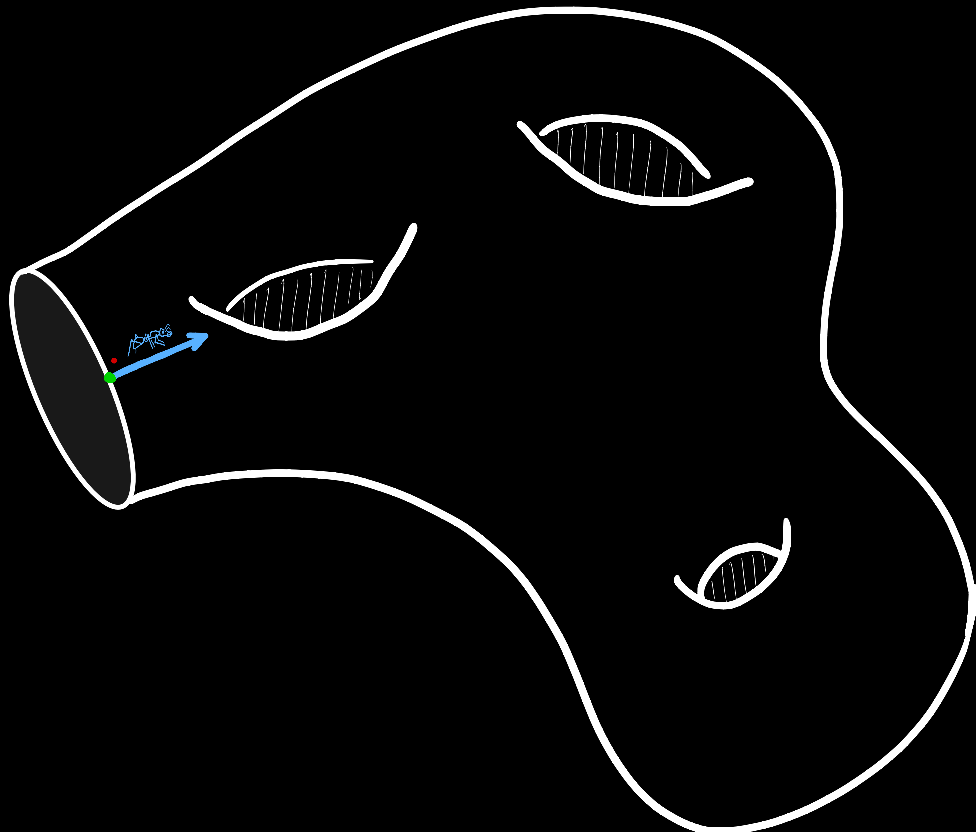
BUDD, BUDZINSKI, C., PETRI 2026+

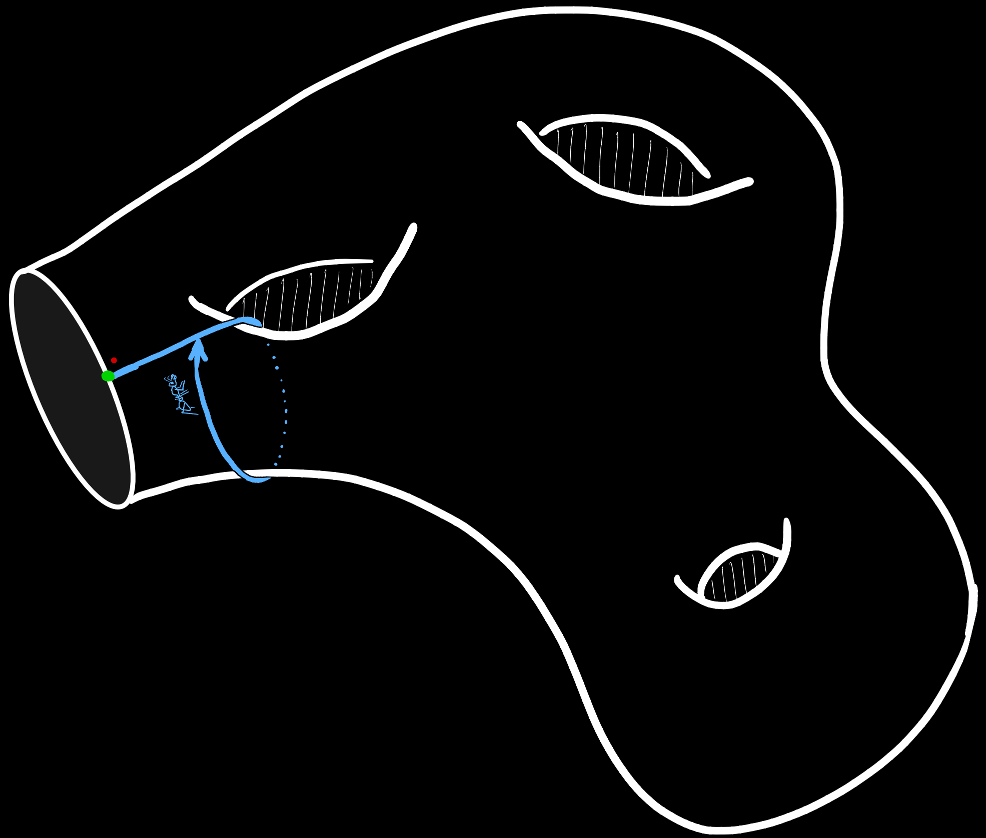
$$\begin{tikzcd} \mathcal{T}_g \arrow[r] & \mathcal{M}_g \arrow[l, bend left, dashed] \end{tikzcd}$$

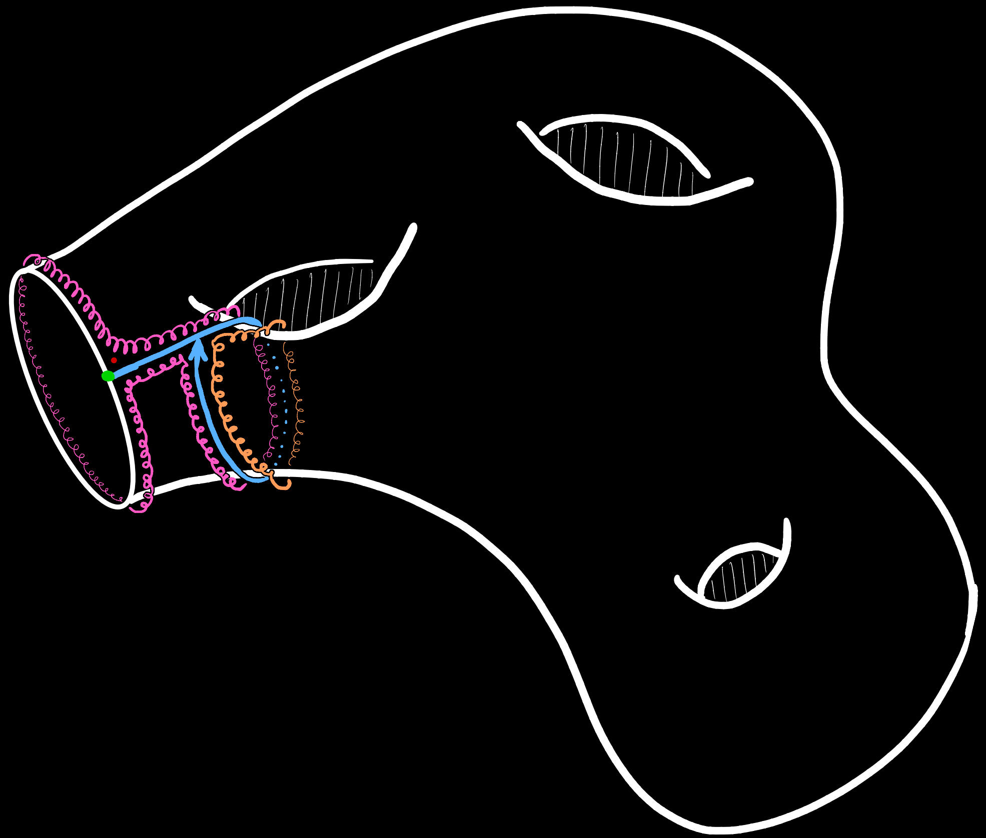
TEICHMÜLLER

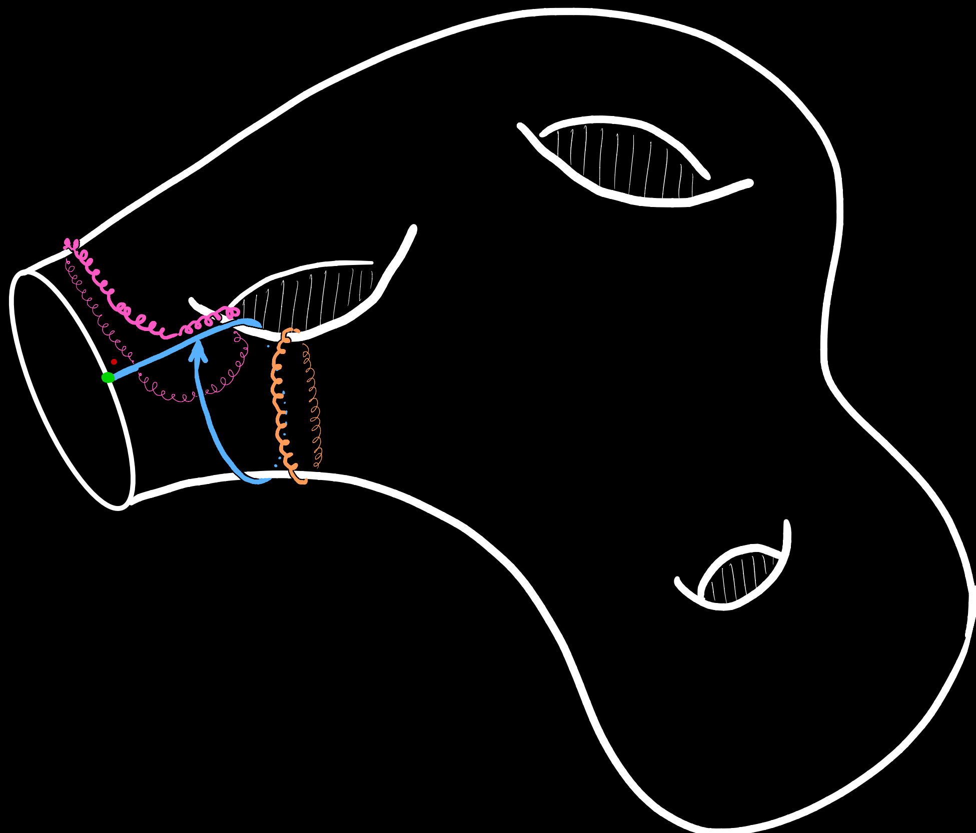
MODULI SPACE

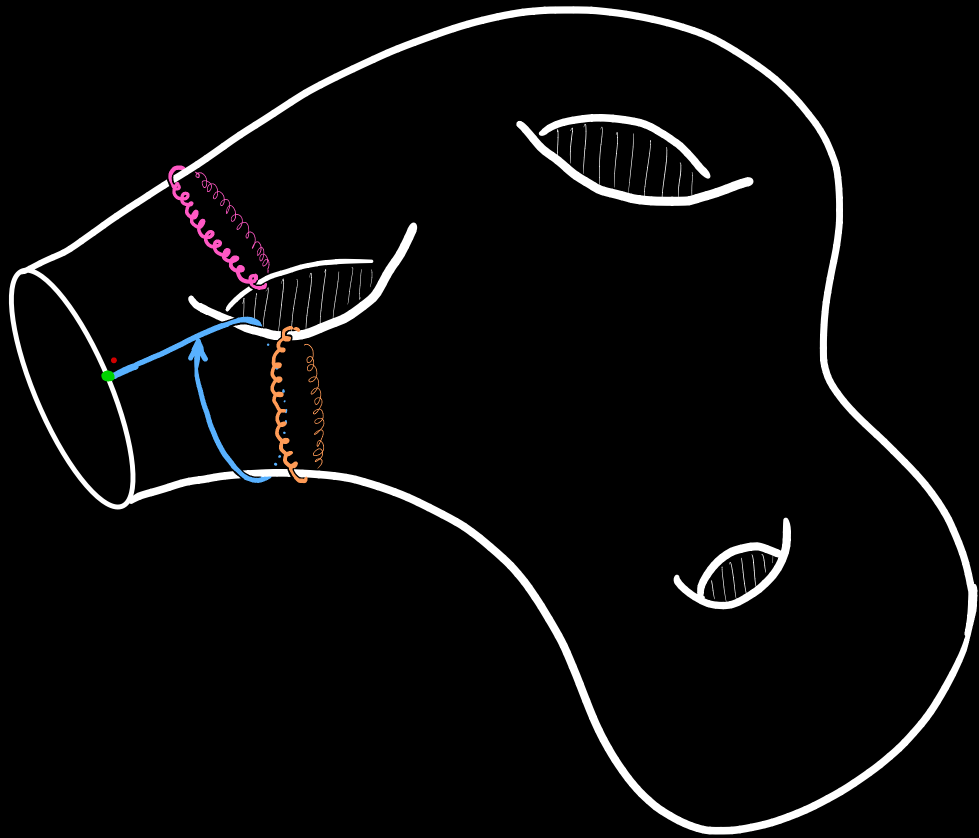
CANONICAL REPRESENTATIVE

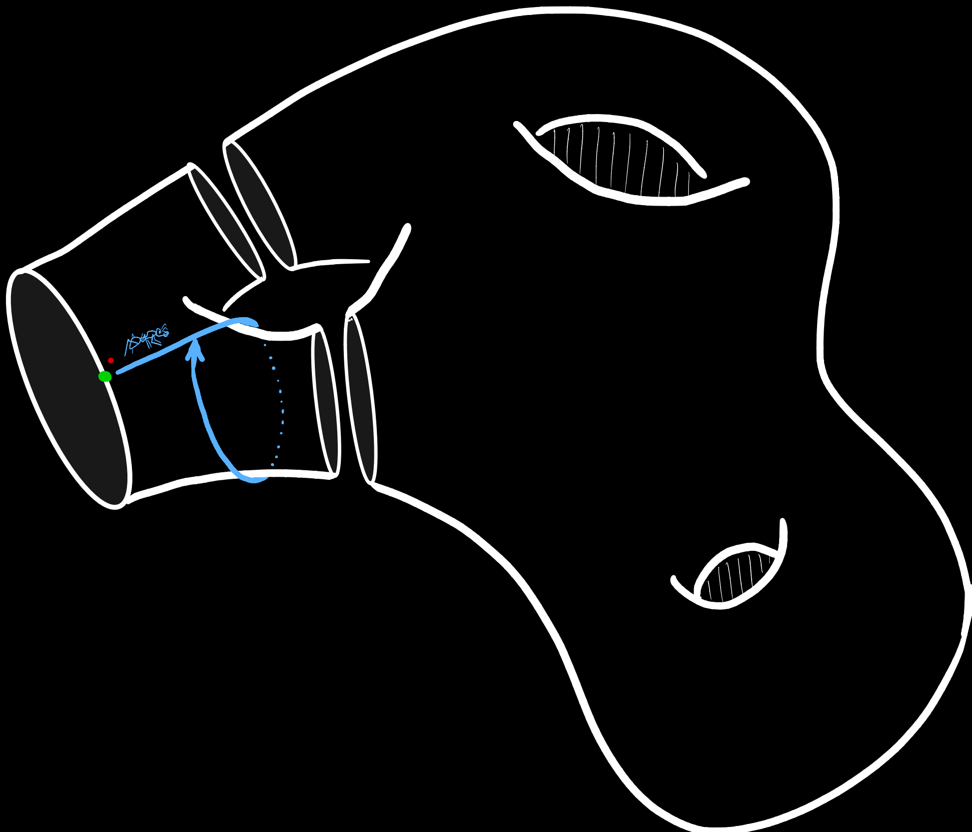


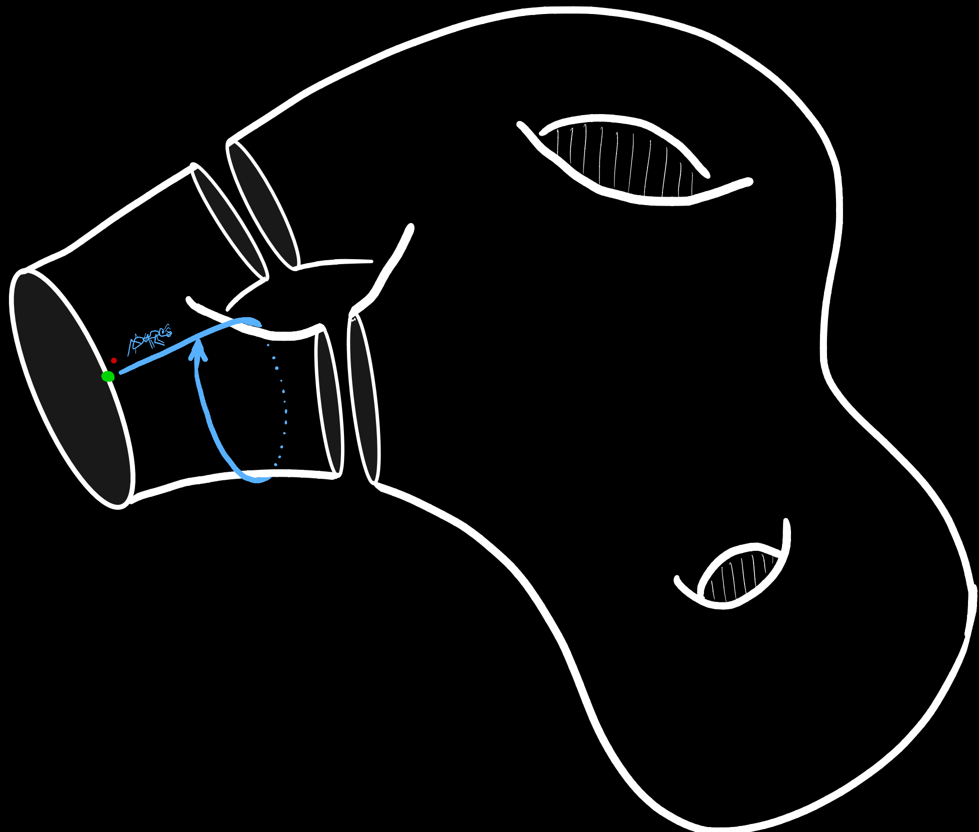


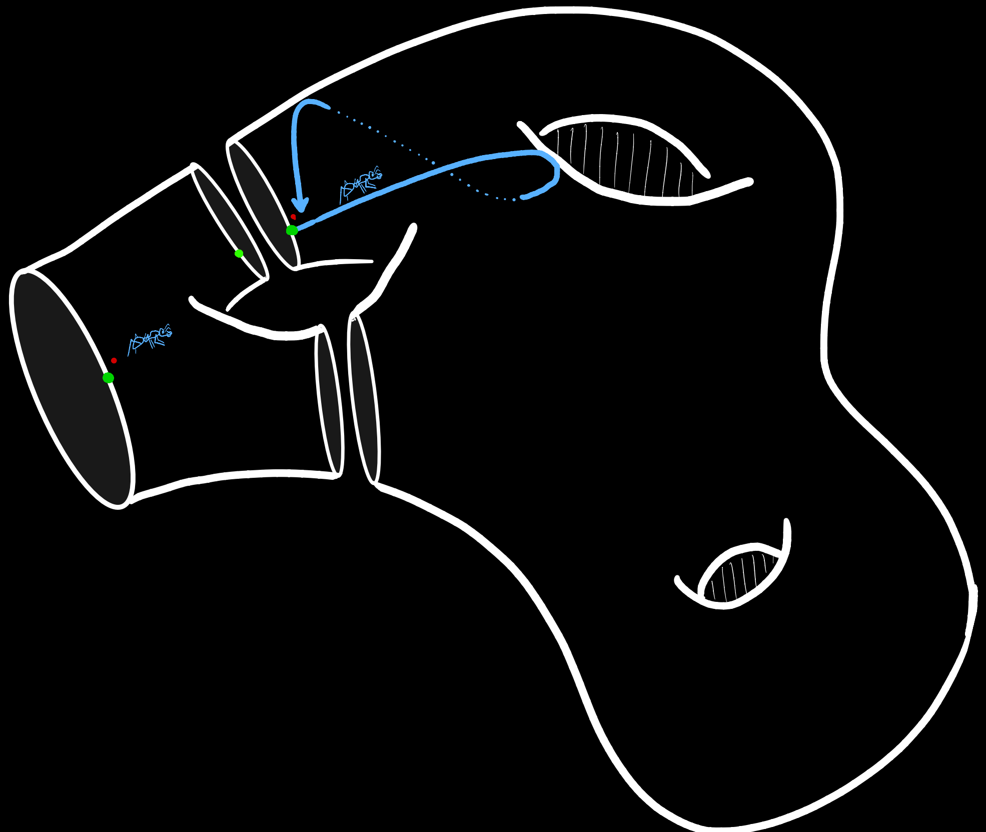


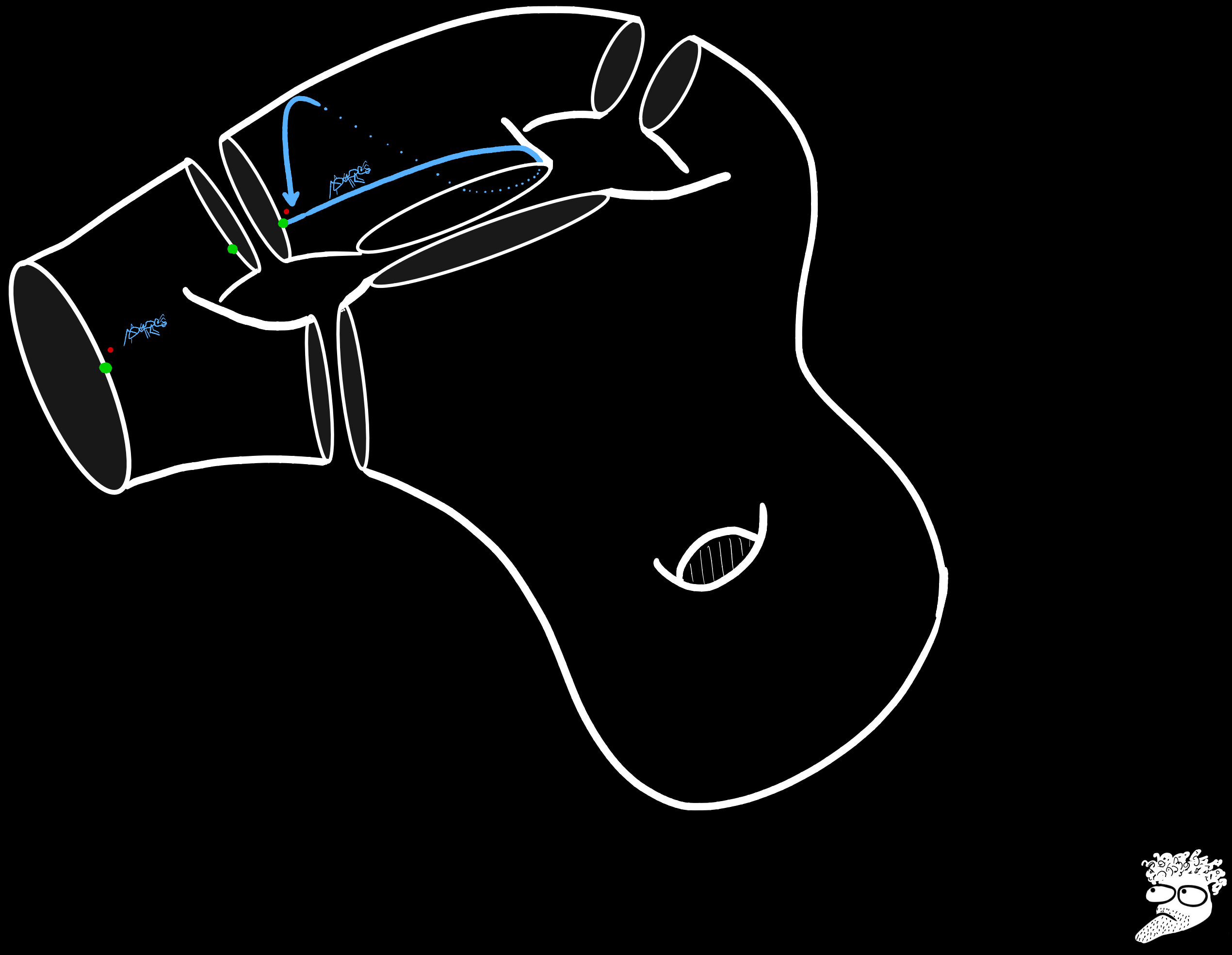


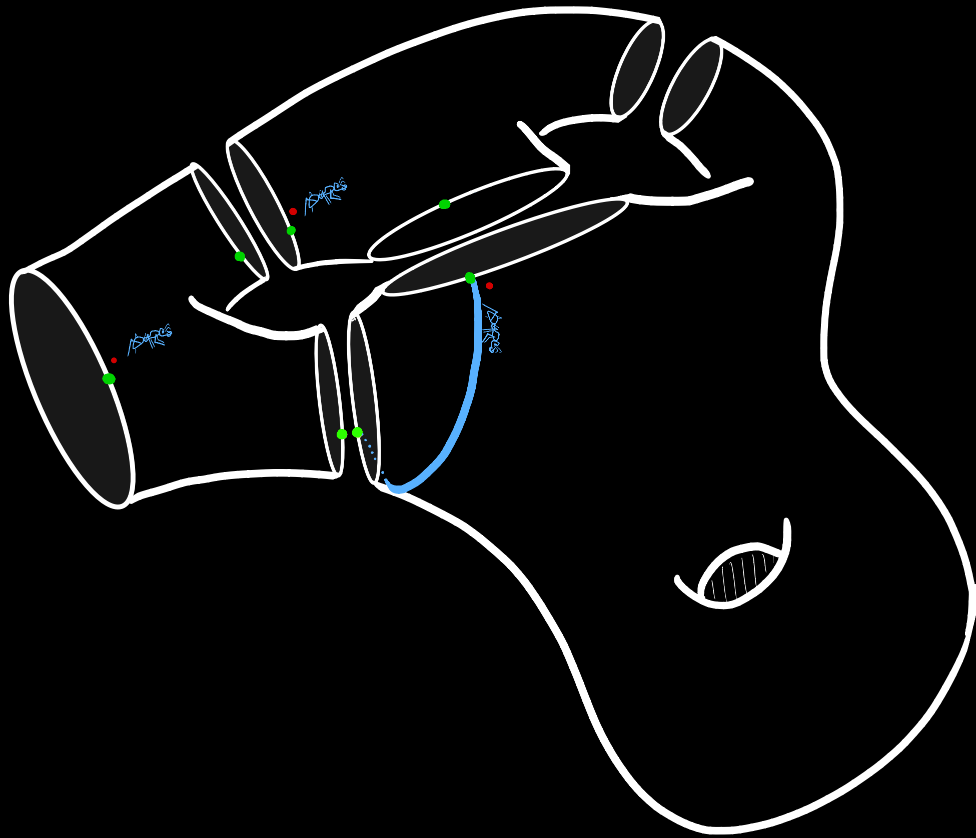


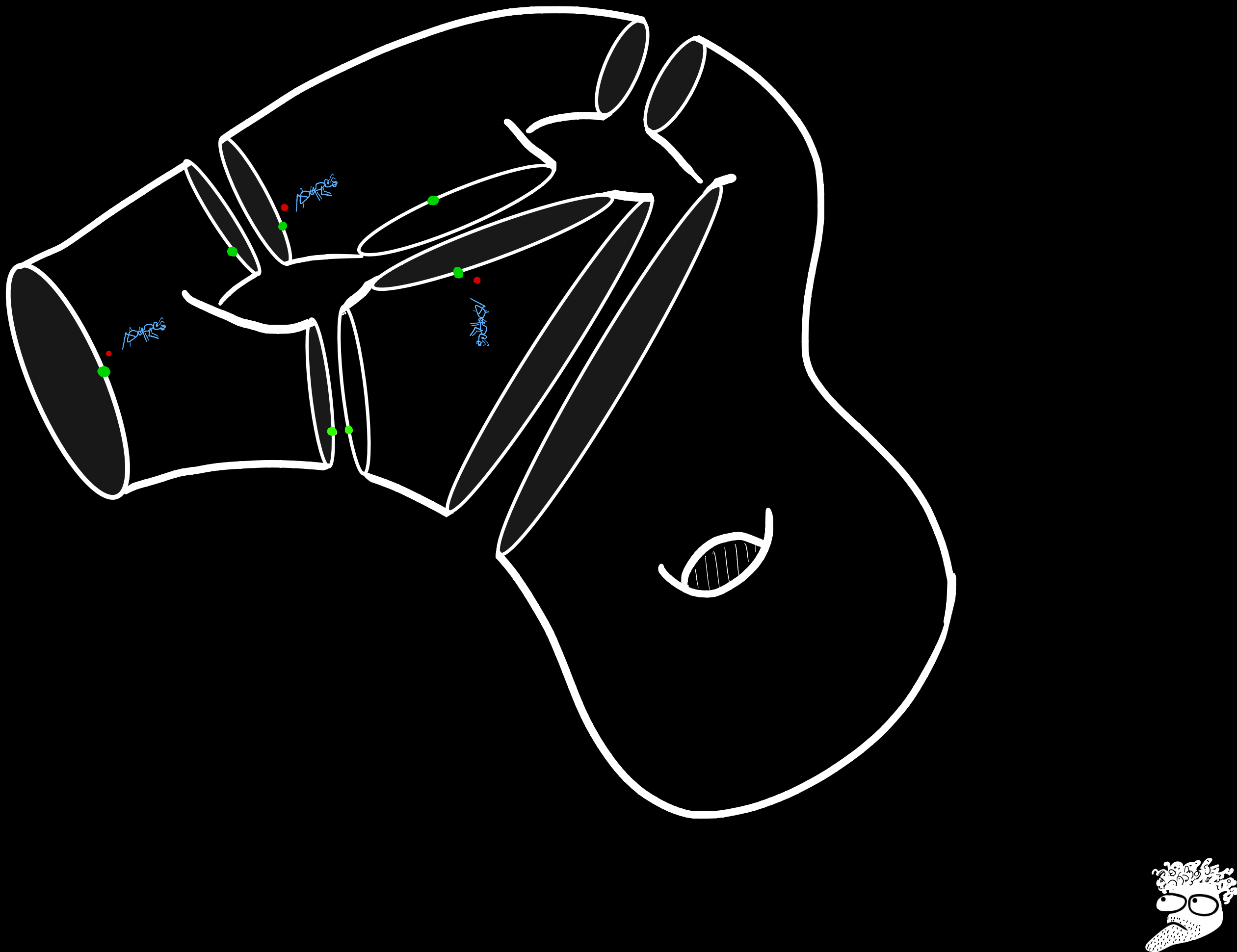


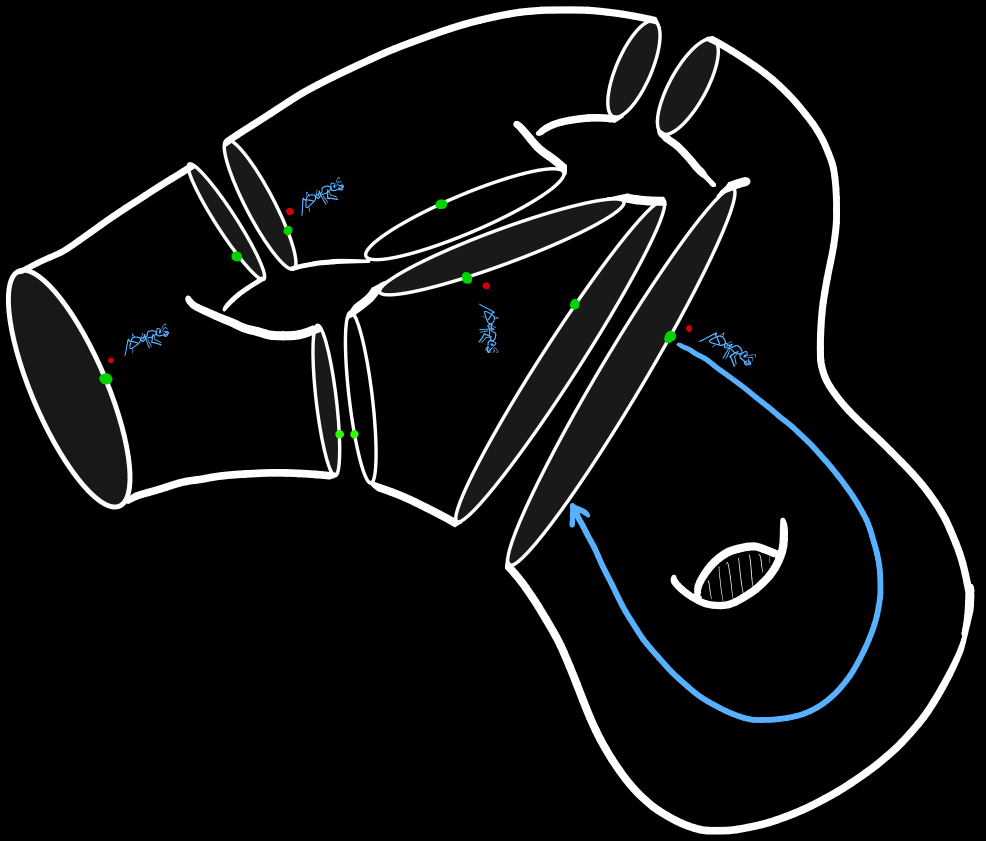


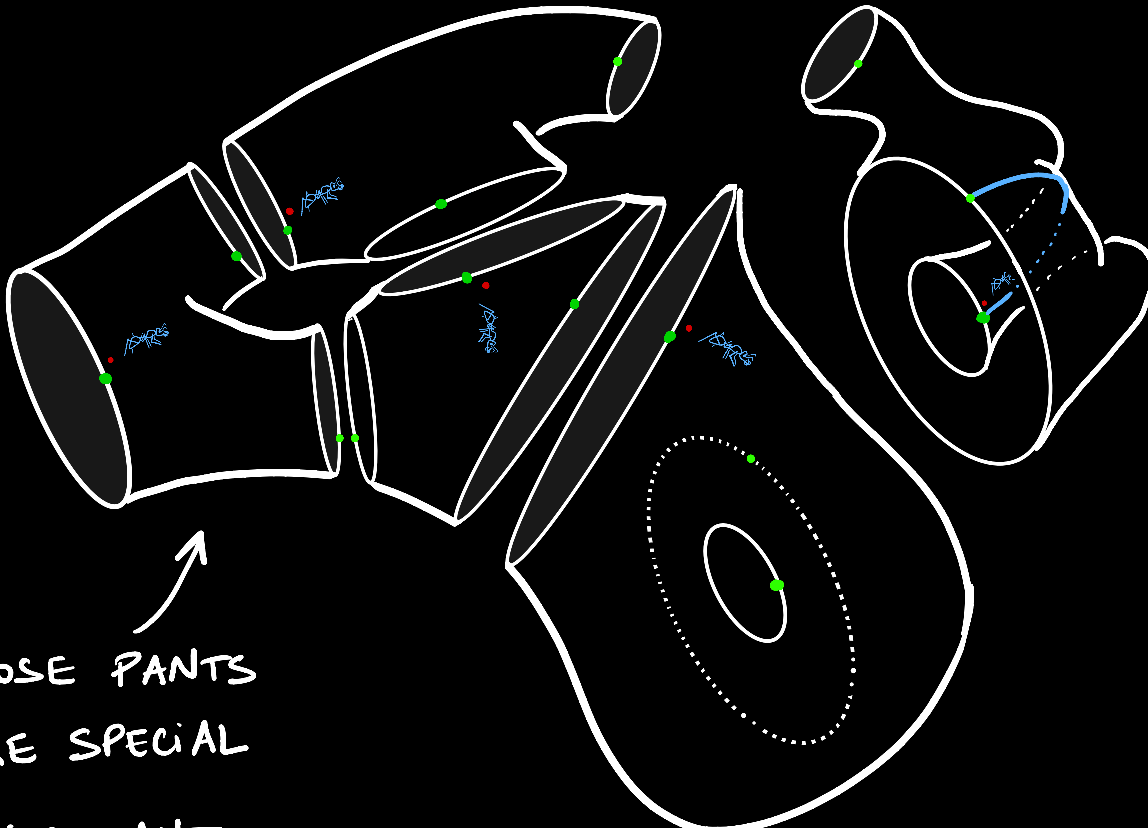












THOSE PANTS ARE SPECIAL

$\underbrace{\text{PANTS + ANT}}$

RANDOM WEIL-PETERSSON HYPERBOLIC SURFACES, HIGH GENUS

SPECTRAL GAP.

ALL BASED ON SELBERG' TRACE FORMULA.

- ANANTHARAMAN-MONK
- WU-XUE
- LIPNOWSKI-WRIGHT
- HIDE-MACERA-THOMAS

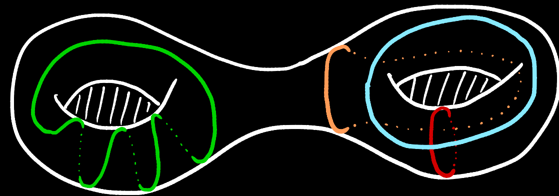
2020-2025

$$\mathbb{P}\left(\lambda_1(S_g) > 1/4 - \varepsilon\right) \xrightarrow[g \to +\infty]{} 0$$

"NEARLY RAMANUJAN / SELBERG SURFACES".

RANDOM WEIL-PETERSSON HYPERBOLIC SURFACES, HIGH GENUS

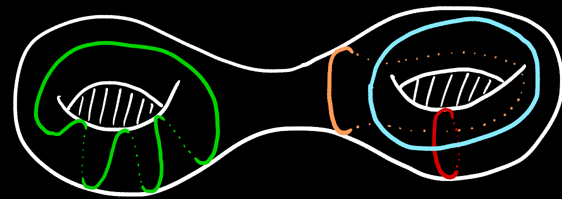
LENGTH SPECTRUM



$\mathcal{L}$ COLLECTION OF LENGTH OF ALL CLOSED GEODESICS

RANDOM WEIL-PETERSSON HYPERBOLIC SURFACES, HIGH GENUS

LENGTH SPECTRUM



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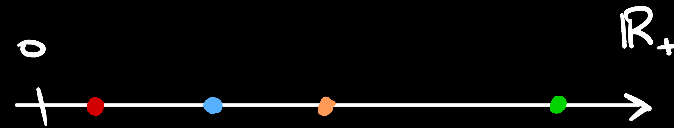
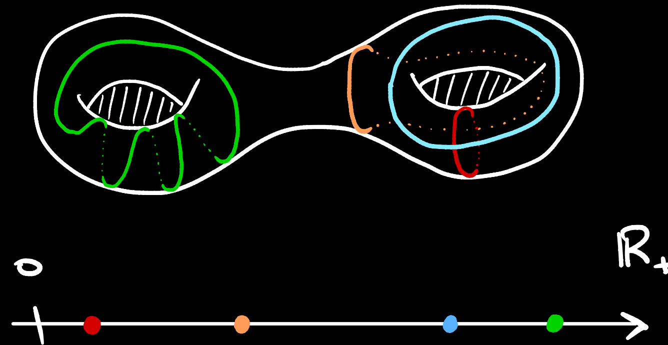


MIRZAKHANI
PETRI
2017'

$$\mathcal{L}(S_g) \xrightarrow{(d)} \text{POISSON POINT PROCESS WITH INTENSITY } \frac{\cosh(t)-1}{t}$$

RANDOM WEIL-PETERSSON HYPERBOLIC SURFACES, HIGH GENUS

LENGTH SPECTRUM



$\mathcal{L}$ COLLECTION OF LENGTH OF ALL CLOSED GEODESICS

MIRZAKHANI
PETRI
2017'

$$\mathcal{L}(S_g) \xrightarrow{(d)} \text{POISSON POINT PROCESS WITH INTENSITY } \frac{\cosh(t)-1}{t}$$

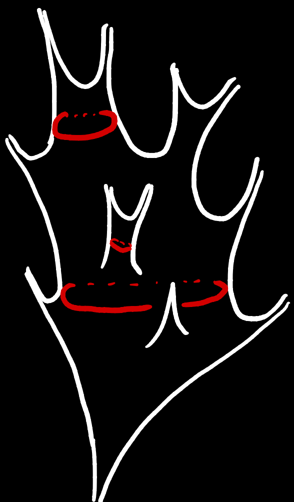
UNIVERSALITY:

- THE SAME AS RANDOM TRIVALENT UNICELLULAR MAPS (JANSON-LOUF)
- ALSO TIGHT LENGTH SPECTRUM (BUDD-LIONS)

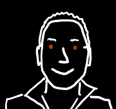
RANDOM WEIL-PETERSSON HYPERBOLIC SURFACES, LOW GENUS

$S_{0,n} \equiv S_n \in \mathcal{M}_{0,n}$ $\quad n \geq 3 \quad$ WP. DISTRIBUTED.

POINTED AT UNIFORM VERTEX.



$$S_n^\bullet$$

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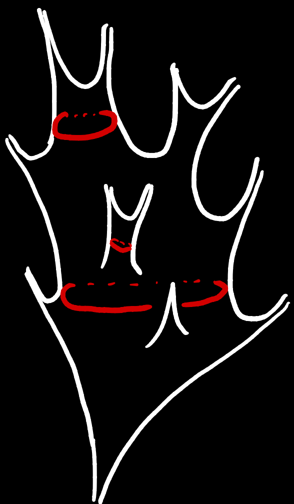
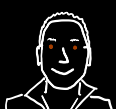


BUDD, C.
2025

WITHOUT RENORM°

$$S_n^\bullet \xrightarrow{n \to +\infty} S_\infty^\bullet$$

INFINITE HYPERBOLIC SURFACE $\simeq \mathbb{R}^2/\mathbb{Z}^2$



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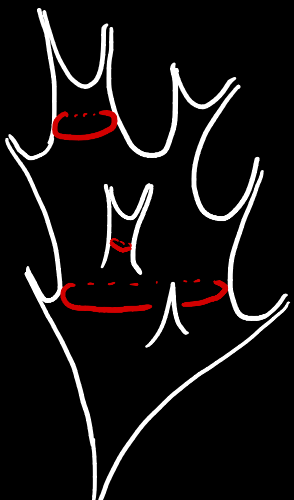


BUDD, C. 2025

$S_n^\bullet$

WITHOUT RENORM°

$n \to +\infty$

NORM $\times\, n^{-1/4}$

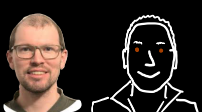
GH / GP

$S_\infty^\bullet$

INFINITE HYPERBOLIC SURFACE $\simeq \mathbb{R}^2/\mathbb{Z}^2$

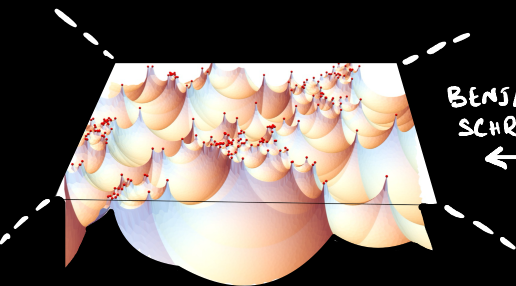
BROWNIAN SPHERE

RANDOM FRACTAL METRIC $\simeq \mathbb{S}^2$



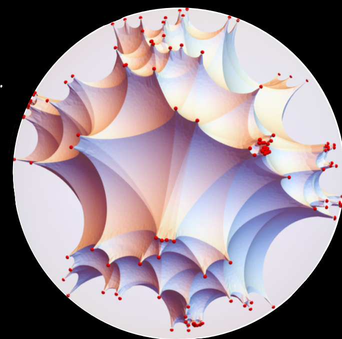
RANDOM WEIL-PETERSSON HYPERBOLIC-SURFACES, LOW GENUS

INFINITE WP PLANE. $\longleftarrow$ $S_n^\bullet$ $\longrightarrow$ BROWNIAN SPHERE

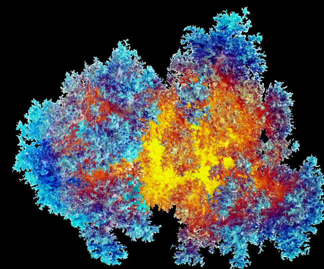


BY T. BUDD

BENJAMINI SCHRAMM $\longleftarrow$



GROMOV PROKHOROV $\longrightarrow$



BY B. STUFLER.

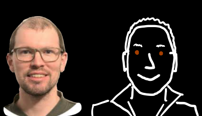
SPECTRAL MEASURE

LOCAL STATISTICS

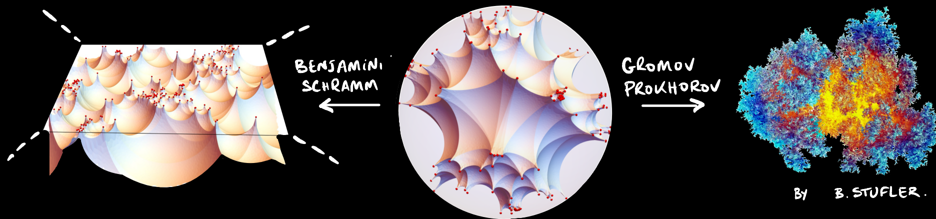
DISCRETE VOLUME GROWTH

DISTANCE BTW K POINTS

LENGTH of SEPARATING GEO



RANDOM WEIL-PETERSSON HYPERBOLIC SURFACES, LOW GENUS

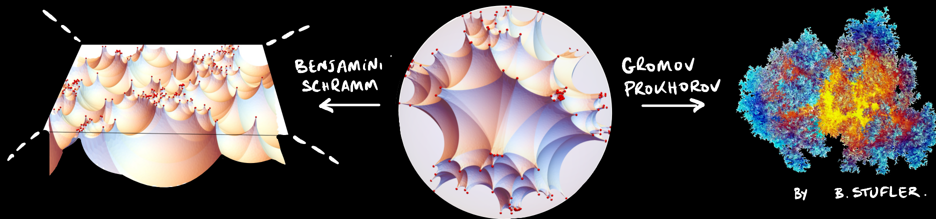


METHODS: → "IN PRINCIPLE" DOABLE WITH PEELING

→ RELY ON BIJECTIONS WITH LABELED TREES

CLEVER SPECIFICATIONS OF PENNER / BOWDITCH-EPSTEIN

RANDOM WEIL-PETERSSON HYPERBOLIC SURFACES, LOW GENUS



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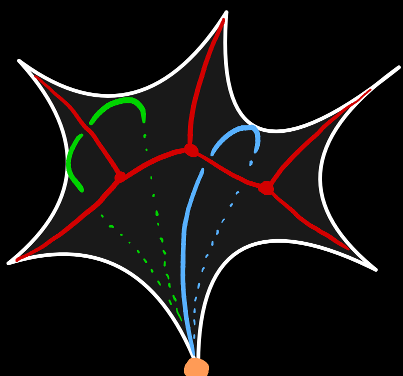
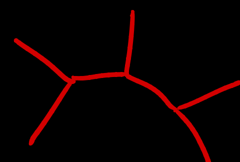


→



+ labels

(distances to ●)

See
BUDD
ZONNENVELD

PERSPECTIVES -

- UNIVERSALITY
  - BROWNIAN SPHERE, STABLE MAPS: LOW GENUS (BUDD ZONNENVELD)
  - LENGTH SPECTRUM, LAPLACIAN SPECTRUM: HIGH GENUS

- DIAMETER WP large g.

- TOWARDS TRACY-WIDOM λ_1

- # GENUS $\simeq$ # PUNCTURES

- OTHER NATURAL MEASURES (THURSTON MEASURE ...)

THANK YOU