Review of probabilistic and discrete models for JT gravity

Frank FERRARI

International Solvay Institutes
Brussels

Probabilistic Paths to QFT
Courant Institute, New York University, NY, USA
April 28th, 2026

References

F. F., Jackiw-Teitelboim Gravity, Random Disks of Constant Curvature, Self-Overlapping Curves and Liouville CFT_1, arXiv:2402.08052, Phys Rev D111 (2025) L061901

F. F., Random Disks of Constant Curvature: the Lattice Story, arXiv:2406.06875, Phys Rev D 113 (2026) 026012

and papers to appear.

Plan of the talk

1. Jackiw-Teiltelboim quantum gravity from physics

2. The Schwarzian field theory

3. Derivation of the Schwarzian field theory from JT

4. Quantum gravity on finite-size geometries and free boundary conditions

5. JT gravity on finite-size geometries: discrete approach, self-overlapping polygons, multiplicity

6. JT gravity on finite geometries: continuum approach, the path integral (probabilistic point of view)

7. JT gravity on finite geometries: continuum approach, quantization: JT CFT

8. From finite size to infinite size: the Schwarzian as a hydrodynamic limit and the emergence of time

The defining feature of Jackiw-Teitelboim quantum gravity (JTQG) is that the bulk curvature is fixed. There are thus three possible models, corresponding to negative curvature, zero curvature or positive curvature. All the models are interesting for physics (holography, cosmology, etc.).

The Schwarzian description only applies to the negative curvature model. The study of the other models thus requires in principle to go beyond the Schwarzian description.

Lagrange multiplier “dilaton” field

$$\begin{aligned}\mathcal{S}[g,\Phi,\zeta] = -\frac{1}{16\pi}\Big[\int_{\mathbb{M}}\Phi\Big(R-2\eta\Big)\mathrm{d}_g x + 2\oint_{\mathrm{S}^1}\Phi k\,\mathrm{d}s\Big] \\ + \frac{\Lambda}{16\pi}\int_{\mathbb{M}}\mathrm{d}_g x + \frac{\lambda}{8\pi}\oint_{\mathrm{S}^1}\mathrm{d}s + \frac{\kappa}{8\pi}\oint_{\mathrm{S}^1}\Phi\mathrm{d}s + S_{\mathrm{mat}}[g,\zeta]\end{aligned}$$

By Gauss-Bonnet, the bulk cosmological constant can be absorbed in a shift of the dilaton in non-zero curvature

On infinite geometries, this action needs to be regularized. On finite geometries, the boundary length is ill-defined and must be renormalized.

$$\mathcal{S}[g,\Phi,\zeta]=-\frac{1}{16\pi}\left[\int_{\mathscr{M}}\Phi\Big(R-\frac{2\eta}{L^2}\Big)\mathrm{d}_g x+2\oint_{\partial\mathscr{M}}\Phi k\,\mathrm{d}s\right]+\frac{\Lambda}{16\pi}\int_{\mathscr{M}}\mathrm{d}_g x+\frac{\lambda}{8\pi}\oint_{\mathrm{S}^1}\mathrm{d}s$$

There are three qualitatively distinct types of boundary conditions that one may impose:

1) Free boundary conditions for the dilaton and the metric. This yields “topological” gravity. The boundaries are geodesics and the moduli space of metric is finite dimensional (old story).
2) “Conformally compact” boundary conditions. This is possible only for the negative curvature model. The geometries are then infinite. The model reduces to the Schwarzian field theory (modern story).
3) New: Dirichlet boundary condition for the dilaton, free boundary condition for the metric. The boundary is then free to fluctuate. The typical geometries have a finite size. Essential to understand finite cut-off holography or the models in zero or positive curvature (new story).

JT gravity from real-world physics: near-extremal black holes

Basic example: Reissner-Nordstrom in 4D (Kerr could also be discussed, etc.)

$$\mathrm{d}s^2 = f(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2 \qquad\qquad f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$$

$$F_{rt} = \frac{iQ}{r^2}$$

$$r_\pm = M \pm \sqrt{M^2 - Q^2}$$

$$T = \frac{1}{4\pi}\left|f'(r_+)\right| = \frac{1}{4\pi r_+^3}\left(r_+^2 - Q^2\right)$$

$$S = \pi r_+^2$$

Extremal: $M = Q = r_0\,, \quad T = 0$

Near horizon region of the extremal black hole: $r - r_0 = \rho \ll r_0$

$$ds^2 = \frac{\rho^2}{r_0^2}\mathrm{d}t^2 + \frac{r_0^2\mathrm{d}\rho^2}{\rho^2} + r_0^2\mathrm{d}\Omega^2 \qquad F_{rt} = \frac{i}{r_0}$$

$$\mathrm{d}s^2 = r_0^2\Big[\frac{\mathrm{d}t^2 + \mathrm{d}z^2}{z^2} + \mathrm{d}\Omega^2\Big] \qquad z = \frac{r_0^2}{\rho}$$

The near-horizon region is $\mathrm{H}_2 \times \mathrm{S}^2$

The geometry develops a long hyperbolic (AdS) throat. By dimensional reduction, on can show that the gravitational dynamics in the throat is governed by the JT action.

The fundamental puzzles

In the near extremal limit, i.e. T goes to zero, one finds that:

$$M = E = r_0 + 2\pi^2 r_0^3 T^2 + \cdots$$

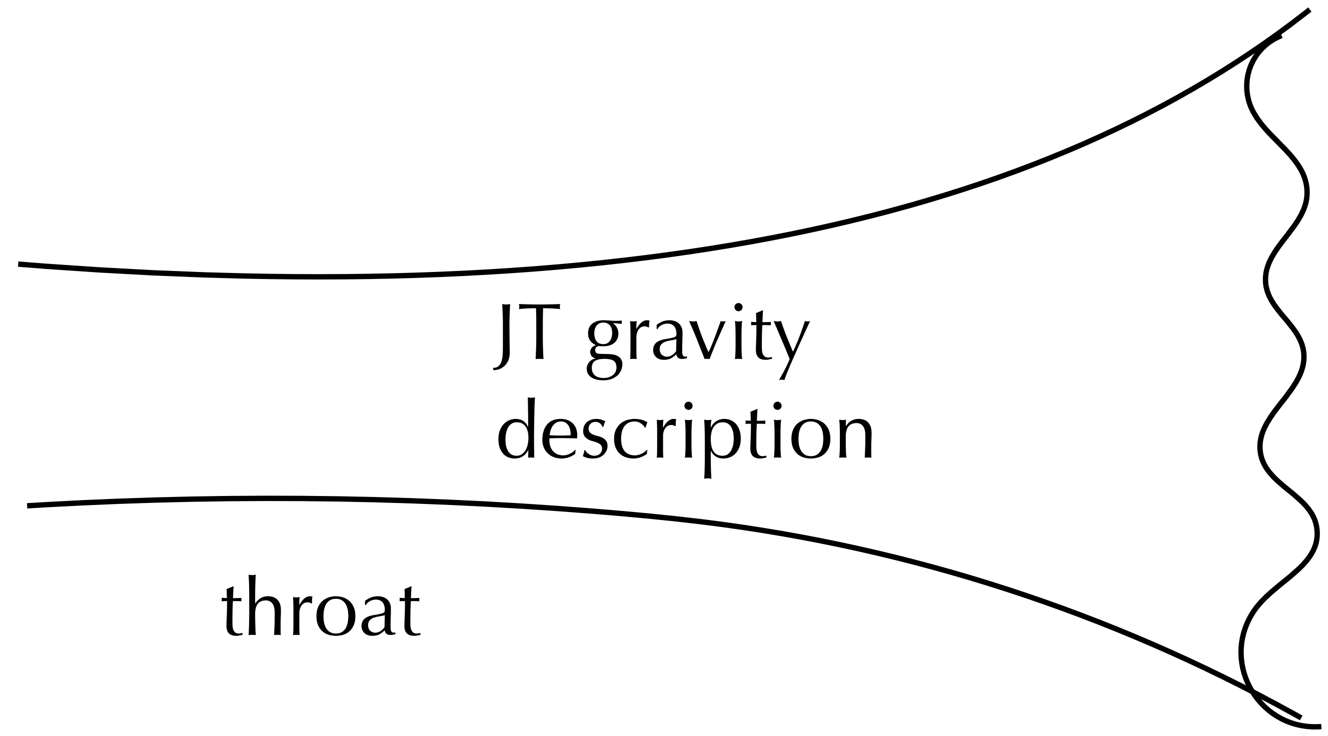
$$S = \pi r_0^2 + 4\pi^2 r_0^3 T + \cdots$$

This semi-classical analysis predicts:

1) A huge ground state degeneracy (extremely suspicious for a non-supersymmetric system)
2) An inconsistency with the usual Hawking evaporation process at as soon as $T \lesssim 1/r_0^3$

The resolution of the puzzle shows that the semi-classical analysis breaks down at very low temperature in spite of the fact that the geometry is arbitrarily weakly curved, due to strong quantum fluctuations of a particular metric mode in the near-horizon region.

This mode is precisely the Schwarzian mode of JT gravity.



horizon

JT gravity
description

throat

Asymptotically
flat region

Smoothly wiggling
boundary described by
the Schwarzian theory

One can take into account these fluctuations exactly. They yield the crucially needed modifications in the low-temperature energy and entropy:

Naive semi-classical:

$$M = E = r_0 + 2\pi^2 r_0^3 T^2 + \cdots \qquad S = \pi r_0^2 + 4\pi^2 r_0^3 T + \cdots$$

Taking into account the Schwarzian mode:

$$M = E = r_0 + \frac{3}{2}T + 2\pi^2 r_0^3 T^2 + \cdots \qquad S = \pi r_0^2 + 4\pi^2 r_0^3 T + \frac{3}{2}\ln T + \cdots$$

The correction to the entropy is associated with a density of state

$$\rho(E) \propto \sinh\left(c\sqrt{E - r_0}\right)$$

The Schwarzian field theory

Configuration space: $\mathrm{Diff}_+(\mathrm{S}^1)/\mathrm{PSL}(2,\mathbb{R})$

$$\frac{1}{\mathrm{Vol}(\mathrm{PSL}(2,\mathbb{R}))}\int D_{\mathrm{L}} f\, e^{-S_{\mathrm{Sch}}[f]}\Big(\cdots\Big)$$

$$S_{\mathrm{Sch}}[f] = -\frac{1}{4\beta_{\mathrm{S}}}\int_0^{2\pi} \mathrm{Sch}\Big[\tan\frac{f}{2}\Big]\,\mathrm{d}\vartheta = \frac{1}{8\beta_{\mathrm{S}}}\int_0^{2\pi}\Big[\Big(\frac{f''}{f'}\Big)^2 - f'^2\Big]\,\mathrm{d}\vartheta$$

$$\mathcal{O}_h(\theta_1,\theta_2) = \left(\frac{f'(\theta_1)f'(\theta_2)}{4\sin^2\frac{f(\theta_1)-f(\theta_2)}{2}}\right)^h$$

Precise definition of the measure: $f'(\theta) = \dfrac{2\pi e^{q(\theta)}}{\int_0^{2\pi} e^{q(\tilde{\theta})}\mathrm{d}\tilde{\theta}}$

$$D_{\mathrm{L}} f\ \exp\Big[-\frac{1}{8\beta_{\mathrm{S}}}\int_0^{2\pi}\Big(\frac{f''}{f'}\Big)^2\mathrm{d}\theta\Big] = Dq\ \exp\Big[-\frac{1}{8\beta_{\mathrm{S}}}\int_0^{2\pi} q'^2\mathrm{d}\theta\Big]$$

Malliavin
Belokurov-Shavgulidze 2017-2019
Bauerschmidt-Losev-Wildemann 2024

Wiener measure

Derivation of the Schwarzian field theory from JT

Why is the Schwarzian field theory a theory of quantum gravity? What is the relation with JT?

The definition of any quantum gravity model in the path integral formalism requires to choose a space of metrics and other fields over which we integrate ("boundary conditions") and to choose a gauge group (an identification between physically indistinguishable configurations).

Appropriate choices in JT gravity yield the Schwarzian description.

I) Space of metrics and fields: conformally compact, asymptotically hyperbolic boundary conditions

$$g = \frac{\bar{g}_a}{a^2} \qquad \Phi = \frac{\bar{\Phi}_a}{a} \qquad a = \text{defining function}$$

$$\bar{g}_{a|\mathrm{S}^1} = h_a \qquad \bar{\Phi}_{a|\mathrm{S}^1} = \varphi_a$$

$$\frac{\varphi_a}{\sqrt{h_a}} = v \quad \text{fixed vector field on the boundary.}$$

$$R = -2 + O(a^2)$$

II) Choice of gauge group: small diffeomorphisms (crucial for the holographic framework)

$$\mathrm{Diff}(\mathbb{M};\mathrm{S}^1)$$

III) Endow M with a collar structure and a choice of a boundary metric (this is an additional external data, which is required to define the renormalized action)

$$\sqrt{h_{\mathrm{b}}}\,\mathrm{d}\theta = e_{\mathrm{b}}\mathrm{d}\theta \qquad (r,\theta) \quad \text{fixed coordinates near the boundary}$$

$$r \text{ is a defining function}$$

On the disk, after integrating out the dilaton, which imposes the constraint R=-2, we are left with the moduli space of metric

$$\mathrm{Met}_{\mathrm{H}}(\mathbb{D})/\,\mathrm{Diff}(\mathbb{D};\mathrm{S}^1) = \mathrm{Diff}_+(\mathrm{S}^1)/\mathrm{PSL}(2,\mathbb{R})$$

Space of conformally compact hyperbolic metrics

Gauge group of small diffeos

Indeed, $\mathrm{Met}_{\mathrm{H}}(\mathbb{D})/\,\mathrm{Diff}(\mathbb{D}) = \{\delta^-\}$

In any orbit [g], there is a unique element which has the Fefferman-Graham form

$$\hat{g} = (f^{-1})^* g = \frac{\mathrm{d}r^2 + (e_\mathrm{b} + e_2 r^2)^2 \mathrm{d}\theta^2}{r^2} \qquad \hat{\Phi} = f \cdot \Phi = \frac{\bar{\Phi}_r}{r} \sim \frac{v e_\mathrm{b}}{r} = \frac{\varphi_\mathrm{b}}{r}$$

The group $\mathrm{Diff}_+(\mathrm{S}^1)$ acts via large diffeos on $\hat{g}$, preserving the FG form.

Under this action, e_b is fixed and e_2 transforms in the coadjoint representation of Virasoro.

Defining the renormalized action

$$\mathcal{S}[g,\Phi,\zeta] = -\frac{1}{16\pi}\Big[\int_{\mathbb{M}} \Phi\Big(R-2\eta\Big)\mathrm{d}_g x + 2\oint_{\mathrm{S}^1} \Phi k \,\mathrm{d}s\Big] + \frac{\Lambda}{16\pi}\int_{\mathbb{M}} \mathrm{d}_g x + \frac{\lambda}{8\pi}\oint_{\mathrm{S}^1} \mathrm{d}s + \frac{\kappa}{8\pi}\oint_{\mathrm{S}^1} \Phi \mathrm{d}s + S_{\mathrm{mat}}[g,\zeta]$$

After integrating out the dilaton, which imposes the constraint R=-2, we are left with an action

$$\mathcal{S}[g] = -\frac{1}{8\pi}\oint_{\mathrm{S}^1} \Phi(k-1)\,\mathrm{d}s$$

Naive regularization: replace $\mathbb{M} \to \mathbb{M}_\epsilon \,, \quad \mathrm{S}^1 \to \mathrm{S}^1_\epsilon \qquad \mathbb{M}_\epsilon : \; r > \epsilon$

$$\mathcal{S}_\epsilon[g] = -\frac{1}{8\pi}\oint_{\mathrm{S}^1_\epsilon} \Phi k \,\mathrm{d}s$$

$$\mathcal{S}_{\mathrm{R}} = \lim_{\epsilon\to 0}\mathcal{S}_\epsilon[\hat{g},\hat{\Phi}] = \frac{1}{4\pi}\int_0^{2\pi} \varphi_{\mathrm{b}} e_2 \,\mathrm{d}\theta$$

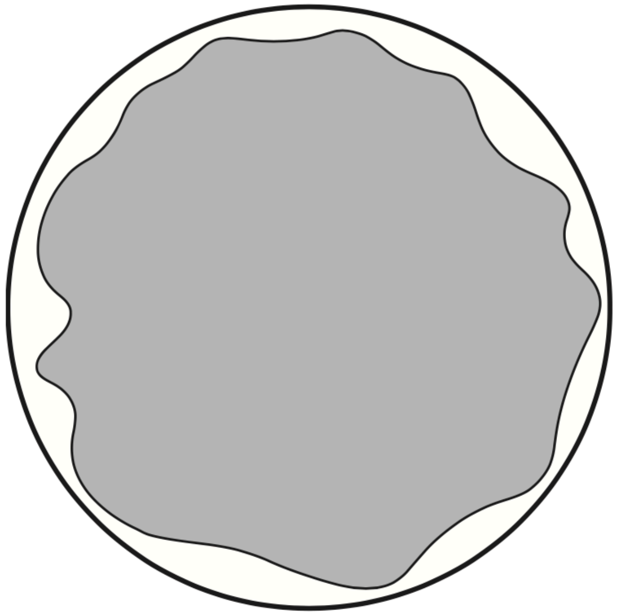
On the disk, with the choice, $e_{\mathrm{b}} = 1$

$$e_2 = -\frac{1}{2}\,\mathrm{Sch}\Big[\tan\frac{f}{2}\Big]$$

(coadjoint orbit containing the constant 1/4)

$$\mathcal{S}_{\mathrm{R}} = -\frac{1}{8\pi}\int_0^{2\pi} \varphi_{\mathrm{b}}\,\mathrm{Sch}\Big[\tan\frac{f}{2}\Big]\,\mathrm{d}\theta \qquad \frac{1}{\beta_{\mathrm{S}}} = \frac{\varphi_{\mathrm{b}}}{2\pi}$$

If one cut-off these metrics at $r = \epsilon$, and represent the resulting finite geometries by embedding the disk isometrically in hyperbolic space, one gets the "smoothly wiggling boundary configurations," of the physics literature, parameterized in the traditional way by the circle diffeomorphism.



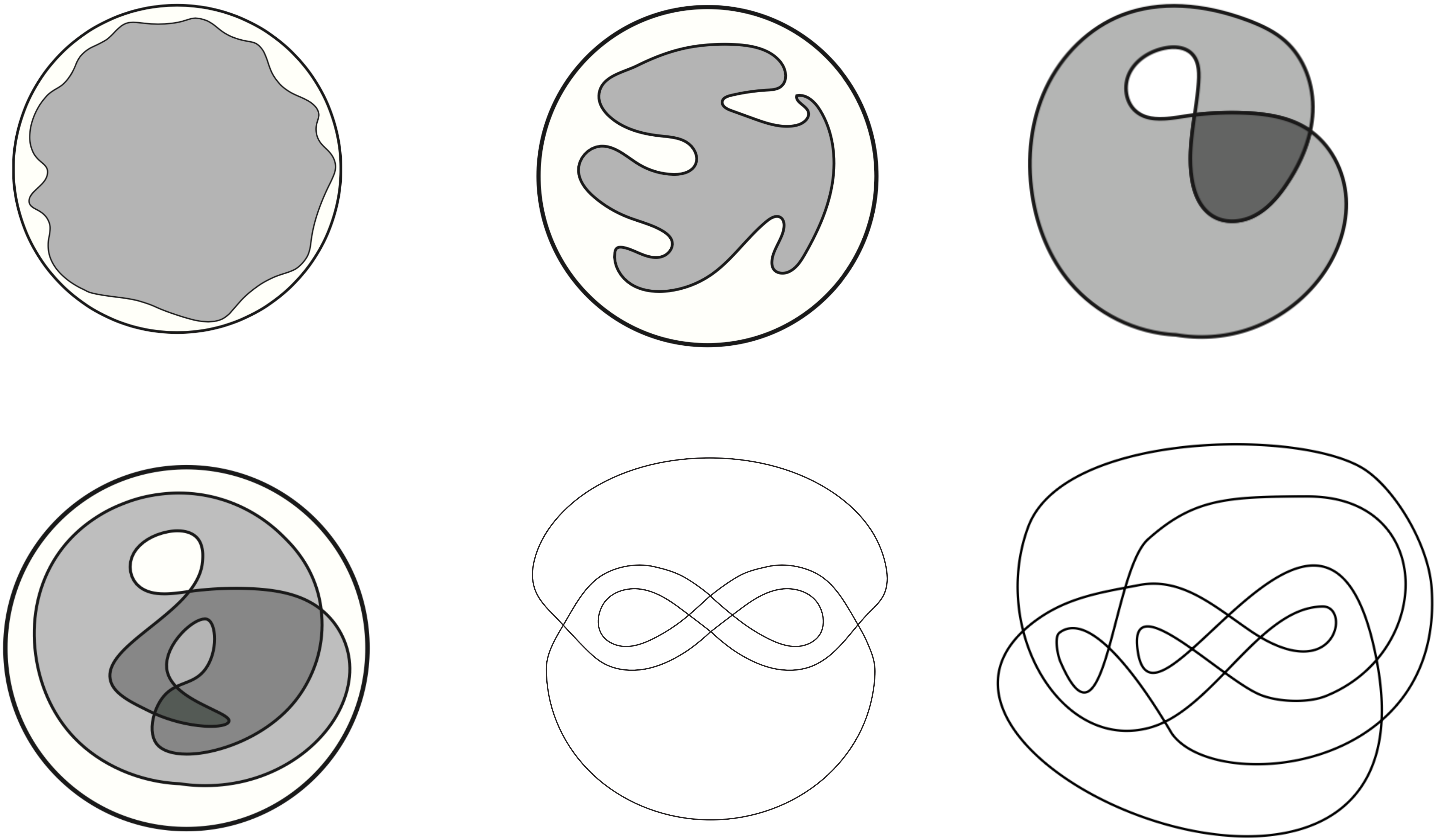
One can also derive the measure, by using the Fadeev-Popov procedure adapted to the case where only small diffeomorphisms are gauged, but we shall skip this step, by lack of time.

JT gravity on finite geometries: space of metrics

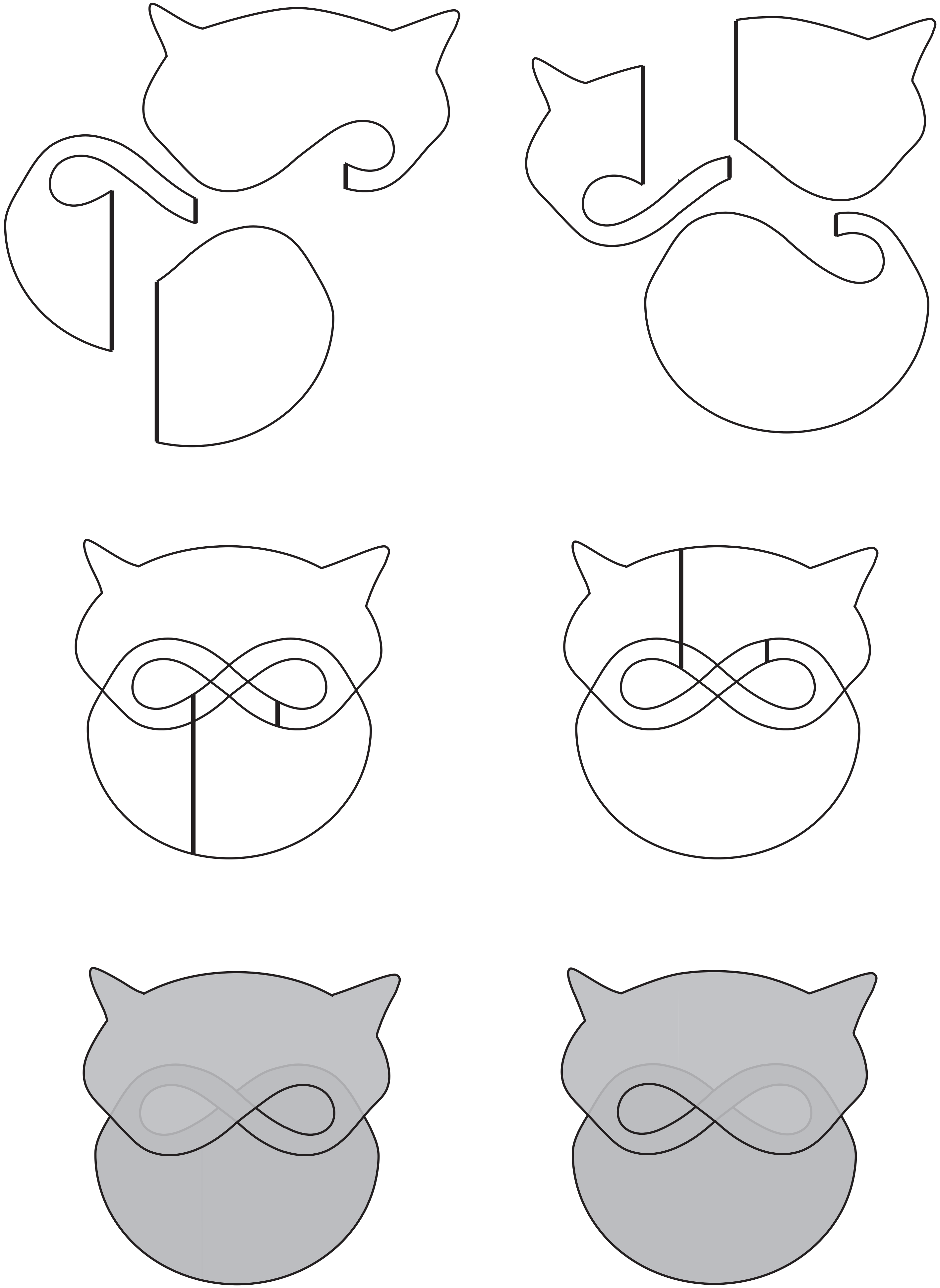
The space of metrics for finite-size geometries (even arbitrarily large) is much larger than the space of conformally compact metrics. This considerable enlargement of the moduli space of metrics drastically changes the physics of the model. Many ideas and concepts used in the usual infinite geometry case are inoperative.

Finite-size geometries (metrics) of constant curvature on the disk can be represented isometrically by immersing the disk in a canonical target space (Euclidean space, hyperbolic space or the round 2-sphere).

Distorted disks

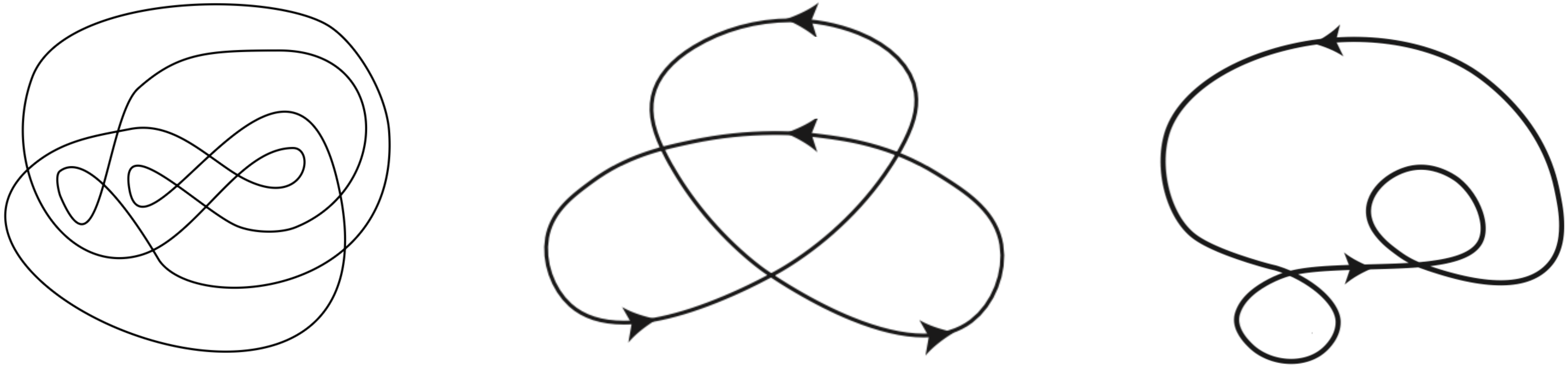


Milnor phenomenon



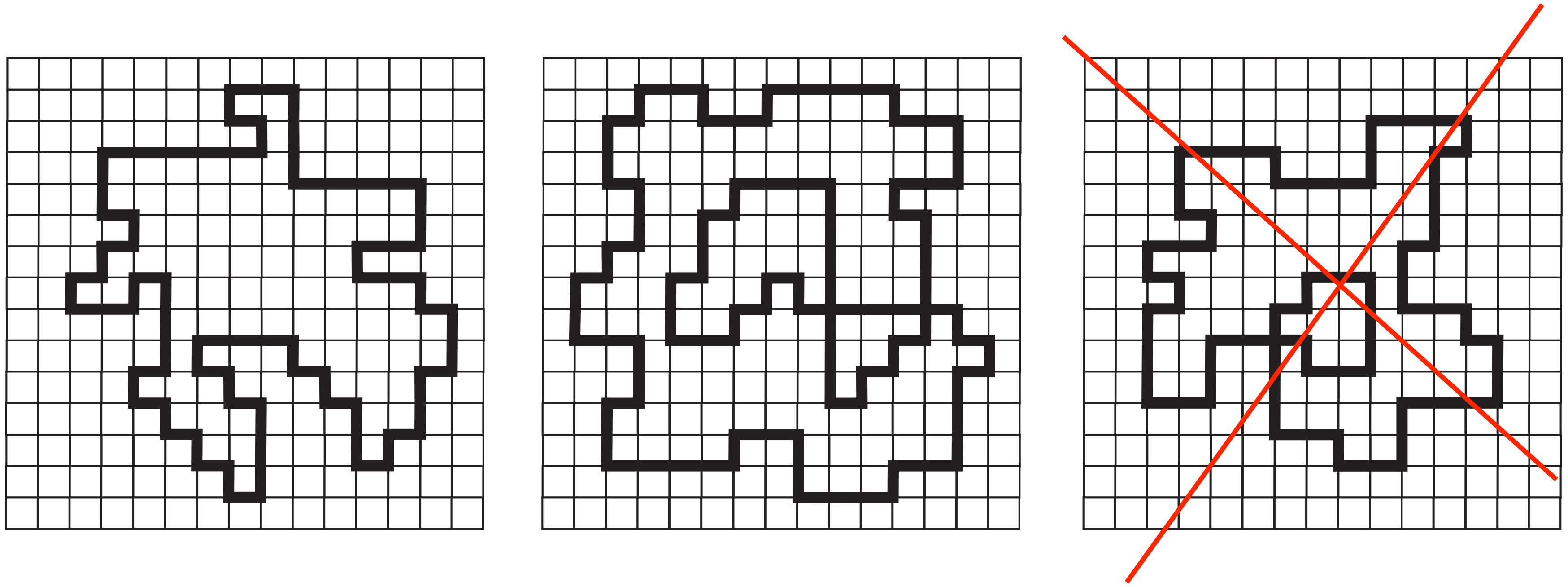
This implies, in particular, that the ``boundary particle picture'' for JT gravity is not correct, since a given boundary curve may bound several distinct distorted disks, which correspond to diffeomorphism-inequivalent constant curvature metrics. A formulation of the model in terms of the boundary curves must therefore use a non-uniform measure, each boundary curve being counted with an appropriate multiplicity.

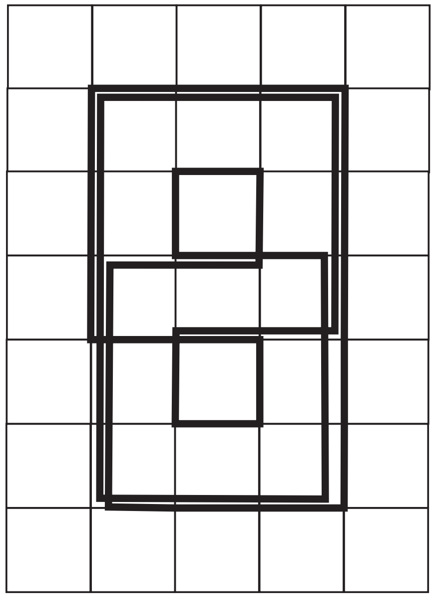
To complicate matters further, a typical arbitrary closed curve is not associated with a metric and therefore should not be included in the configuration space of the theory. The allowed closed curves are called self-overlapping loops. There exists polynomial-time algorithms to decide whether a closed curve is self-overlapping or not and to compute the associated multiplicity.

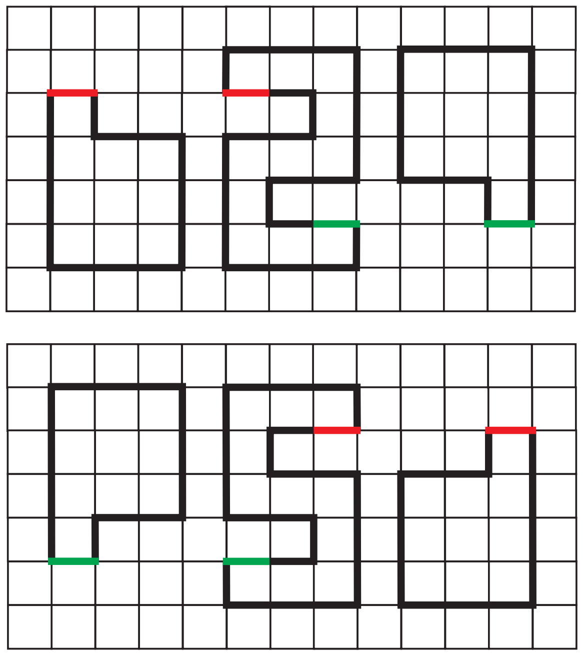


JT gravity on finite geometries: discretized formulation

JT gravity is a random curve model of a new type. We have to count self-overlapping curves, taking into account the multiplicity.







$$2n = 48\,, \qquad p = 31$$

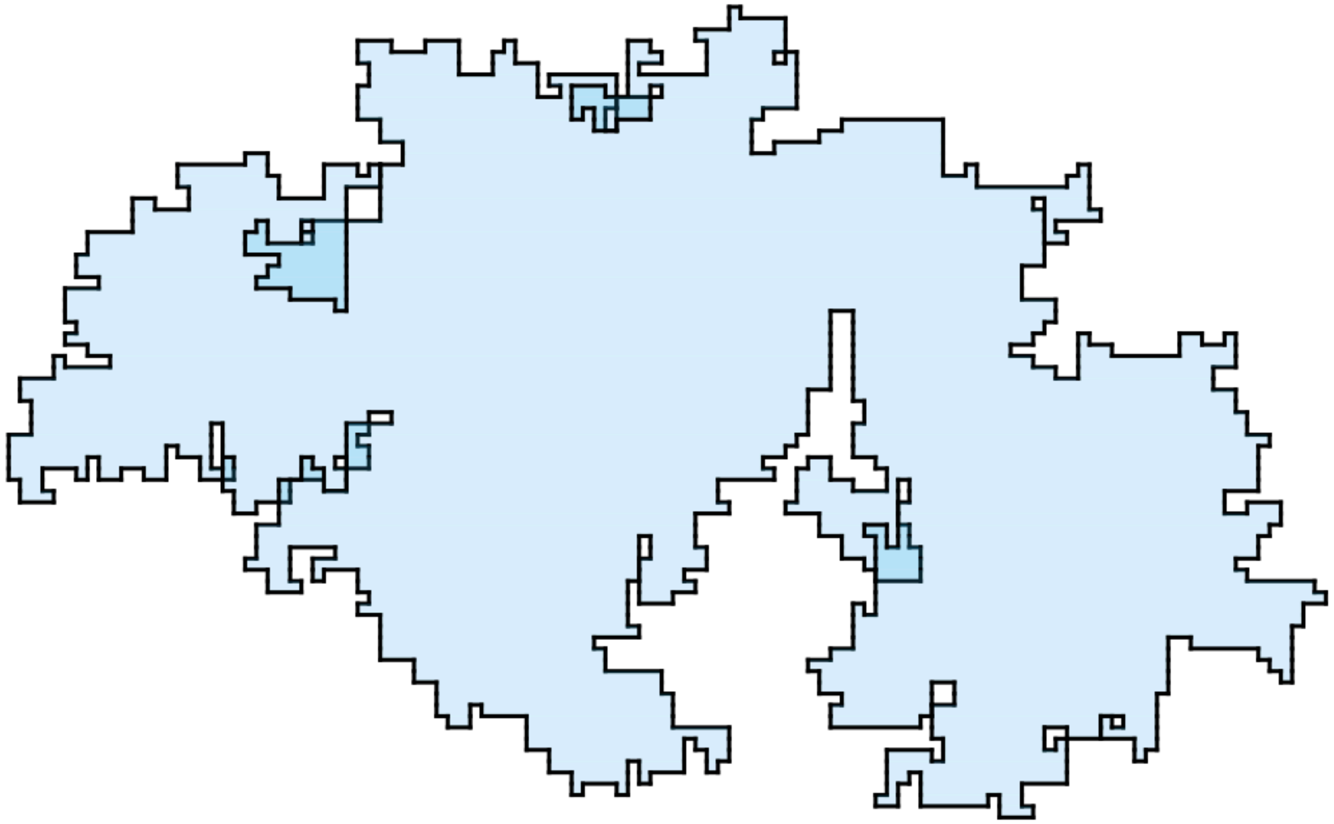
The combinatorial problem has a matrix model formulation, using the idea of dually weighted graphs (Di Francesco and Itzykson).

exponential growth
proven in Ferrari 2024

Log-correction
predicted in Ferrari
2024, confirmed in
Budd/Zonneveld 2026

$$\text{number of configurations} \sim e^{an} n^{-3} (\ln n)^{b}$$

Power law behaviour
predicted in Ferrari
2024, confirmed in
Budd/Zonneveld 2026



One important conceptual point is that the discretized approach, based on polygonization of surfaces, always implicitly assumes that both small and large diffeomorphisms are gauged. Indeed, the combinatorial data encoded in a given polygonization are purely metric, both in the bulk and in the boundary. Thus, they provide no information on privileged coordinate systems, either in the bulk or on the boundary.

A direct consequence of this fact is that any continuum formulation of the theory, that has any chance of being equivalent to the discretized formulation, must necessarily by formulated by gauging both small and large diffeomorphisms.

$$\mathrm{Diff}(\mathbb{M};\mathrm{S}^1):\ \text{infinite geometries with asymptotic boundaries}$$

$$\mathrm{Diff}(\bar{\mathbb{M}}):\ \text{finite-size geometries}$$

Important physical consequences: there is no Hamiltonian or absolute notion of time when quantum gravity is formulated on finite-size geometries, even when a boundary is present.

JT gravity on finite geometries: path integral

A crucial insight is to realize that the difficult combinatorial problem of summing over all finite geometries in JT gravity can be formulated in a suprisingly simple an elegant way, directly in the continuum.

A key point is that, when the large diffeomorphisms are gauged, it is always possible to go to conformal gauge. Techniques that were originally developed for perturbative string theory and Liouville gravity then become accessible. This yields a JT gravity path integral and a conformal field theory description that shares many similarities with the familiar Liouville path integral and Liouville CFT.

A fundamental conjecture is that this continuum description is equivalent to the continuum limit of the discretized version of the theory.

In spirit, this idea bears some similarities with the SLE story. The SLE processes describe the universal conformally invariant continuum limit of self-avoiding random curve or loop models, which includes many physically relevant cases, for instance the random self-avoiding loops counted with a uniform measure, critical Ising model interfaces, critical percolation interfaces, level lines of random surfaces, etc. The magic of SLE is to relate these complicated-looking, highly non-Markovian processes, to Brownian motion; in the appropriate parameterization, the relevant measure is simply the Wiener measure.

A similar simplification occurs for the self-overlapping random loops relevant to JT gravity. The associated process is not the Brownian motion and the correct measure is slightly more complicated than the Wiener measure, but it can be constructed rigorously in terms of known ingredients, specifically the one-dimensional log-correlated Gaussian free field. The description for the flat JT model is disconcertingly simple, given the complexity of the constrained space of metrics one needs to take into account.

Let us describe the result for the flat model, on the disk topology.

$$\mathscr{D} = \Big\{ z \in \mathbb{C} \mid |z| < 1 \Big\}$$

$$\mathrm{d}s^2 = e^{2\phi/Q} |\mathrm{d}z|^2 \qquad\qquad Q \geq 2$$

Flatness implies that ϕ is harmonic. It is thus entirely determined by its boundary value.

$$\varphi(\theta) = \sum_{n \in \mathbb{Z}} \varphi_n e^{in\theta}\,, \quad \varphi_n^* = \varphi_{-n}$$

$$\phi(\rho, \theta) = \sum_{n \in \mathbb{Z}} \varphi_n \rho^{|n|} e^{in\theta}$$

Note that the bulk metric is always smooth, because a harmonic function is automatically smooth. This smoothness of the bulk geometry is always true in JT gravity, in sharp contrast with Liouville gravity, as a consequence of the constant bulk curvature constraint.

The metric is made random by using the path integral measure

$$D\varphi\, e^{-\mathcal{S}[\varphi]} = \mathrm{d}\varphi_0 \prod_{n\geq 1} (\mathrm{d}\,\mathrm{Re}\,\varphi_n \mathrm{d}\,\mathrm{Im}\,\varphi_n)\, e^{-\mathcal{S}[\varphi]}$$

$$\mathcal{S} = S_0 + S_\Lambda$$

$$S_0 = \frac{1}{4\pi}\int_0^{2\pi} \Big(\varphi \partial_\rho \phi(\theta, \rho = 1) + 2Q\varphi\Big)\mathrm{d}\theta = \sum_{n\geq 1} n|\varphi_n|^2 + Q\varphi_0$$

(governs the UV behaviour)

$$S_\Lambda = \frac{\Lambda}{16\pi}\int_{\mathscr{D}} e^{2\phi/Q}\,\mathrm{d}^2 z$$

(cosmological constant term)

The measure is PSL(2,R) invariant and the whole formalism is background-independent (see below).

Note that, because the bulk field ϕ is smooth, the area operator $e^{2\phi/Q}$ is not renormalized (it's a naturally protected operator). On the other hand, the boundary Liouville field is a very irregular, distribution-valued field, implying that the boundary is a fractal.

A renormalized ``quantum'' boundary length β can be defined by suitably renormalizing the boundary length operator, exactly as in the case of the Liouville theory,

$$\beta = \int_0^{2\pi} :e^{\frac{\gamma}{2}\varphi}: \mathrm{d}\theta$$

$$\gamma = Q - \sqrt{Q^2 - 4}\,, \quad Q = \frac{2}{\gamma} + \frac{\gamma}{2}$$

The quantum length has an anomalous dimension, β^ν having the dimension of a classical length, with a critical exponent ν, $1/2 \leq \nu \leq 1$, given by

$$\nu = \frac{2}{\gamma Q}$$

The Hausdorff dimension of the boundary is

$$d_{\mathrm{H}} = \frac{1}{\nu} = \frac{\gamma Q}{2} = 1 + \frac{\gamma^2}{4}$$

When JT gravity is coupled to a matter conformal field theory of central charge c,

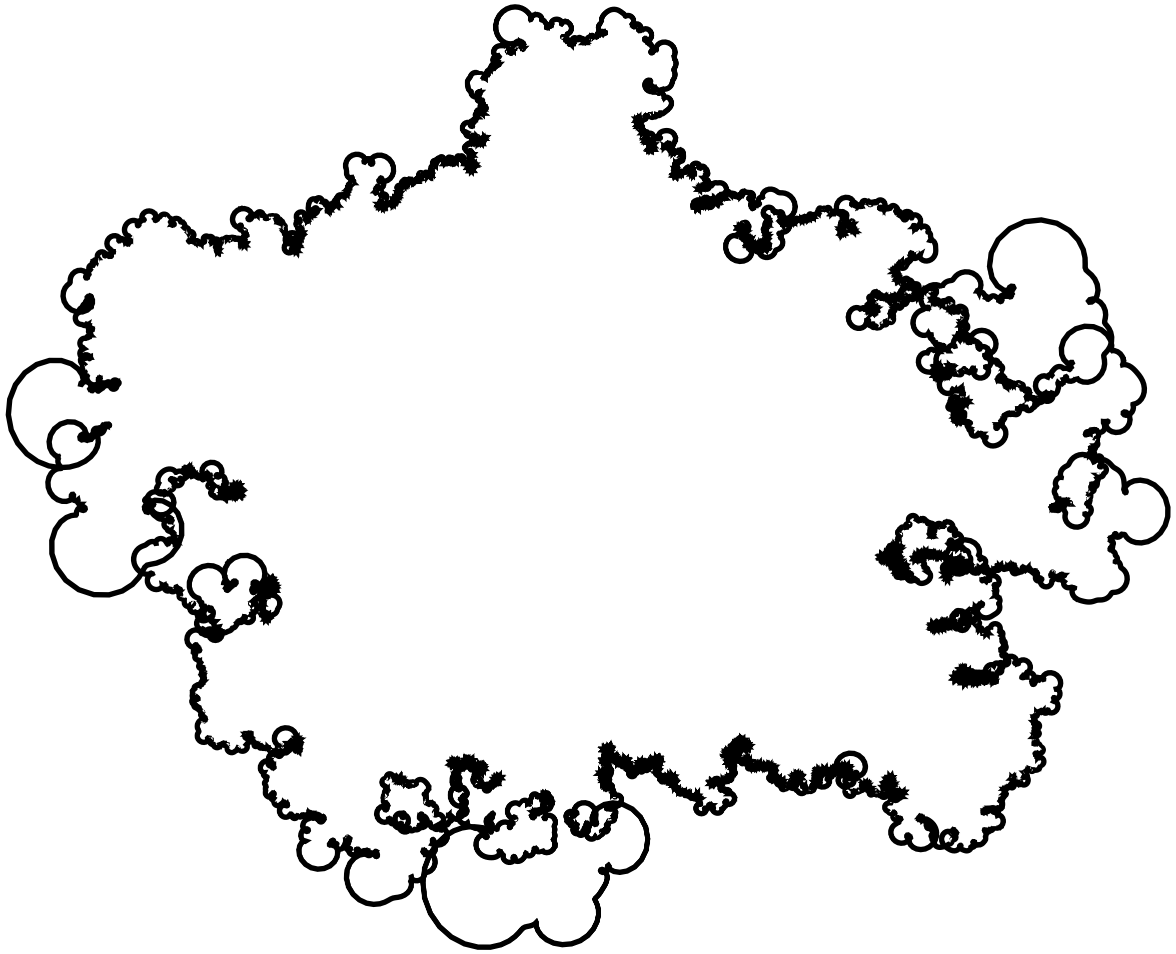
$$Q^2 = \frac{24 - c}{6} \qquad \nu = \frac{1}{2}\left[1 + \sqrt{\frac{c}{c-24}}\right]$$

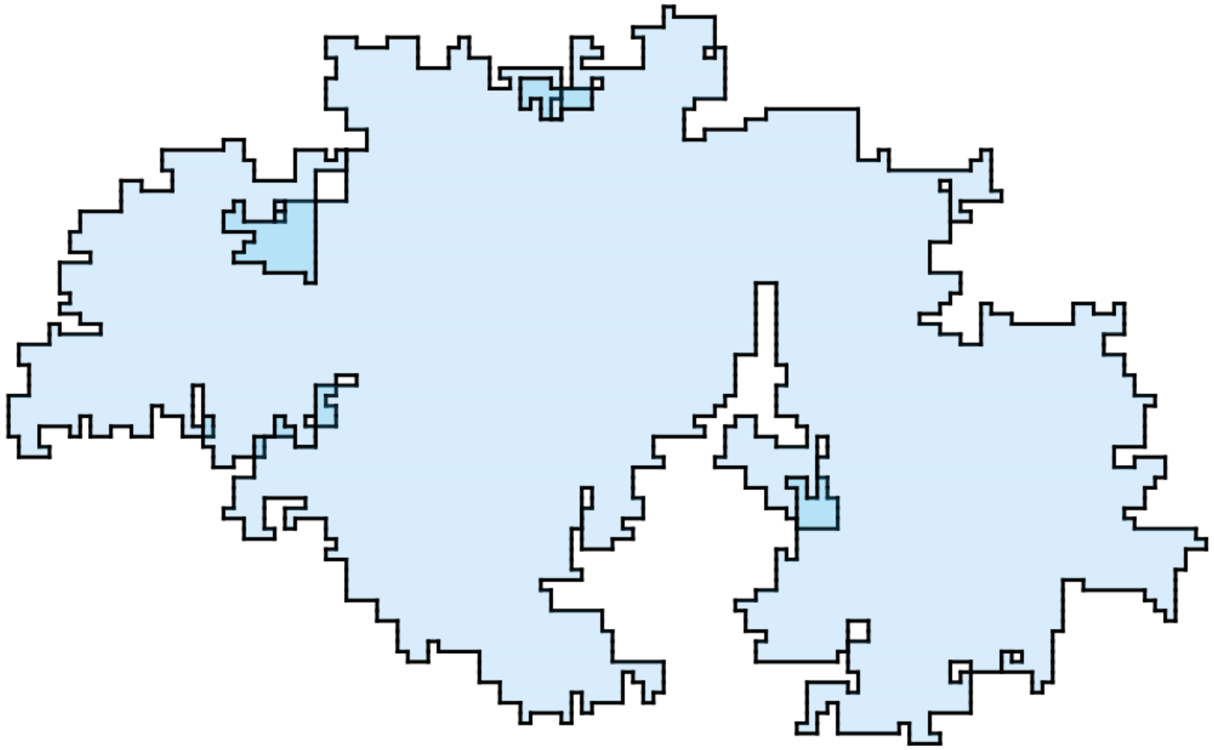
Classical limit (Chaudhuri and Ferrari): $\gamma \to 0$ $Q \to +\infty$ $d_{\mathrm{H}} \to 1$

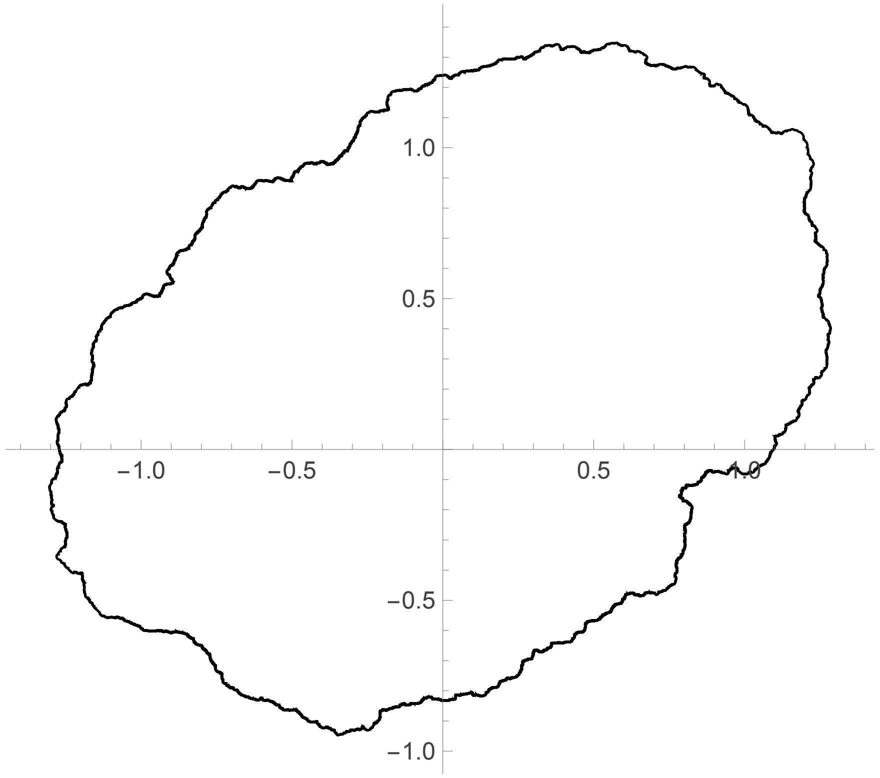
$c \to -\infty$

Pure JT: $\gamma = Q = 2$ $d_{\mathrm{H}} = 2$ $c = 0$

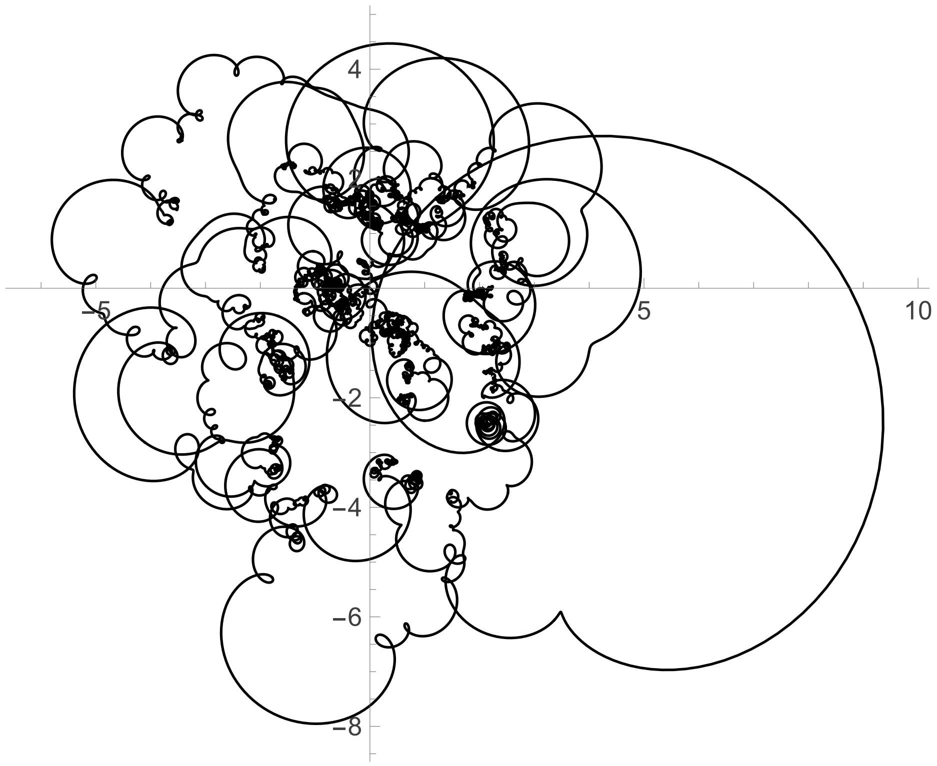
Pure JT is similar to the c=1 Liouville theory (in some sense, in the same universality class).







$c = -125$



$c = 12$

Argument

Using large diffeomorphisms, any metric can be put in the form

$$g = e^{2\Sigma} g_0(\lambda)$$

Note that using large diffeos is crucial in order to be able to go to conformal gauge.

The path integral measure over unconstrained metrics then reads

$$D_{\mathrm{WP}}\lambda \, D_g\Sigma \left[\frac{\det(P_1^\dagger P_1)}{\underbrace{\det\langle \xi_a^{(0)} | \xi_b^{(0)} \rangle_g \det\langle \psi_i^{(0)} | \psi_j^{(0)} \rangle_g}_{e^{-\frac{26}{24\pi} S_{\mathrm{L}}(g_0,\Sigma)}}} \right]^{1/2} \underbrace{Z_{\mathrm{CFT}}[g]}_{e^{\frac{c}{24\pi} S_{\mathrm{L}}(g_0,\Sigma)}} \left(\cdots \right)$$

$$S_{\mathrm{L}}(g_0, \Sigma) = \int_{\mathscr{M}} \mathrm{d}^2x \sqrt{g_0} \left(g_0^{\mu\nu} \partial_\mu \Sigma \partial_\nu \Sigma + R_0 \Sigma \right) + 2 \int_{\partial\mathscr{M}} \mathrm{d}s_0 \, k_0 \Sigma$$

$$D_{\mathrm{WP}}\lambda\,\det(P_1^\dagger P_1)_0\,D_g\Sigma\,e^{\frac{c-26}{24\pi}S_{\mathrm{L}}(g_0,\Sigma)}$$

$$||\delta\Sigma||^2=\int(\delta\Sigma)^2\mathrm{d}_gx=\int(\delta\Sigma)^2e^{2\Sigma}\mathrm{d}_{g_0}x$$

String theory: c=26, the 2D gravitational dynamics decouple.

Liouville theory:

$$\int D_g\Sigma\,e^{\frac{c-26}{24\pi}S_{\mathrm{L}}(g_0,\Sigma)}\longrightarrow\int D_{g_0}\Sigma\,e^{\frac{c-25}{24\pi}S_{\mathrm{L}}(g_0,\Sigma)}$$

``Miracle'': the right-hand side is background independent. This is equivalent to the vanishing of the total central charge.

JT gravity / topological gravity:

We impose the constraint of constant curvature, inserting

$$\delta(R[g] - 2\eta)$$

in the path integral (this factor comes from integrating out the Lagrange multiplier dilaton field). We thus impose

$$R[g] = 2\eta = e^{-2\Sigma}(R[g_0] + 2\Delta_0\Sigma)$$

For negative or zero curvature, a fundamental theorem states that the solution to this equation for the Liouville field is uniquely determined by the boundary value of the field.

For instance, for zero curvature and a flat background metric, the constraint is simply that the Liouville field is harmonic, $\Delta\Sigma = 0\,.$

The degree of freedom we are left with is thus the boundary Liouville field

$$\sigma = \Sigma_{|\partial\mathscr{D}}$$

In the bulk, $\Sigma(x) = \Sigma(x;\sigma)$

Expanding in Fourier modes

$$\Sigma = \sum_{n\in\mathbb{Z}} \sigma_n \rho^{|n|} e^{in\theta} \quad \text{if} \quad \sigma = \sum_{n\in\mathbb{Z}} \sigma_n e^{in\theta}$$

We can perform the path integral over the bulk Liouville field by decomposing

$$\Sigma(x) = \Sigma_{\mathrm{D}}(x) + \Sigma(x;\sigma)$$

The $\delta(R[g] - 2\eta)$ function term set $\Sigma_{\mathrm{D}}(x) = 0$ and produce a $\dfrac{1}{\det(\Delta - R)}$ factor. We get

$$D_{\mathrm{WP}}\lambda \, D_g\sigma \left[\frac{\det(P_1^\dagger P_1)}{\det\langle \xi_a^{(0)} | \xi_b^{(0)} \rangle_g \det\langle \psi_i^{(0)} | \psi_j^{(0)} \rangle_g}\right]^{1/2} \frac{1}{\det(\Delta - R)} Z_{\mathrm{CFT}}[g] \left(\cdots\right)$$

we integrate over the boundary Liouville field only

$$||\delta\sigma||^2 = \oint \delta\sigma^2 \mathrm{d}s = \oint \delta\sigma^2 e^{\sigma} \mathrm{d}s_0$$

$$R[g] = 2\eta = e^{-2\Sigma}(R[g_0] + 2\Delta_0\Sigma)$$

Topological gravity (geodesic boundaries):

$$\sigma = 0 \qquad\qquad (\det(P_1^\dagger P_1))^{1/2} = \det(\Delta - R)$$

JT gravity: $R[g] = 2\eta = e^{-2\Sigma}(R[g_0] + 2\Delta_0\Sigma)$

We have a fluctuating boundary Liouville field.

Let us restrict ourselves to the flat theory, for simplicity.

$$D_{\mathrm{WP}}\lambda\, D_g\sigma \left[\frac{\det(P_1^\dagger P_1)}{\det\langle\xi_a^{(0)}|\xi_b^{(0)}\rangle_g \det\langle\psi_i^{(0)}|\psi_j^{(0)}\rangle_g}\right]^{1/2} \frac{1}{\det\Delta} Z_{\mathrm{CFT}}[g]\left(\cdots\right)$$

$$\uparrow \qquad\qquad\qquad \uparrow \qquad\qquad\qquad \uparrow$$

$$e^{-\frac{26}{24\pi}S_{\mathrm{L}}(g_0,\Sigma)} \qquad e^{\frac{2}{24\pi}S_{\mathrm{L}}(g_0,\Sigma)} \qquad e^{\frac{c}{24\pi}S_{\mathrm{L}}(g_0,\Sigma)}$$

$$D_{\mathrm{WP}}\lambda \, \frac{\det(P_1^\dagger P_1)_0}{\det \Delta_0} \, D_g\sigma \, e^{\frac{c-24}{24\pi} S_{\mathrm{L}}(g_0,\Sigma[\sigma])}$$

$$||\delta\sigma||^2 = \oint \delta\sigma^2 \mathrm{d}s = \oint \delta\sigma^2 e^{\sigma} \mathrm{d}s_0$$

Same idea as in the case of Liouville. Since we do not know how to make sense of measures like $D_g\sigma$ (and they actually probably do not make any sense), replace it by something well-defined

$$D_g\sigma \longrightarrow D_{g_0}\sigma$$

But the resulting theory must be background independent, and this is no longer manifest.

$$e^{\frac{c-24}{24\pi} S_{\mathrm{L}}(g_0,\Sigma[\sigma])} \longrightarrow e^{\frac{?}{24\pi} S_{\mathrm{L}}(g_0,\Sigma[\sigma])}$$

The background independence with respect to changes of background metrics that do not change the boundary Liouville field is manifest.

It is straightforward to show that we can always use such a change to pick a flat background metric.

$$R_0 = 0$$

The bulk Liouville field is then simply the harmonic extension of the boundary Liouville field,

$$\Sigma[\sigma] = H[\sigma]$$

The non-trivial part is to check the background independence with respect to flat background metrics,

$$g_0 \to e^{2\phi} g_0 \qquad\qquad \phi \text{ harmonic}$$

By using $R_0 = 0$ and integrating by part, one can check that the Liouville action takes a very nice form

$$S_{\mathrm{L}}(g_0, \Sigma) = \int_{\mathscr{M}} \mathrm{d}^2 x \sqrt{g_0} \left(g_0^{\mu\nu} \partial_\mu \Sigma \partial_\nu \Sigma + R_0 \Sigma \right) + 2 \int_{\partial \mathscr{M}} \mathrm{d}s_0 \, k_0 \Sigma$$

$$S_{\mathrm{L}} = \oint (\sigma \mathscr{O} \sigma + 2 k_0 \sigma) \mathrm{d}s_0$$

$$\mathscr{O}\sigma = \partial_{n_0} H[\sigma]$$

$\mathscr{O}\sigma = \partial_{n_0} H[\sigma]$ is the Dirichlet-to-Neumann operator.

$$\frac{24 - c}{24\pi} S_{\mathrm{L}} = S_0 \qquad \phi = Q\Sigma \qquad g_0 = |\mathrm{d}z|^2$$

$$S_0 = \frac{1}{4\pi} \int_0^{2\pi} \Big(\varphi \partial_\rho \phi(\theta, \rho = 1) + 2Q\varphi \Big) \mathrm{d}\theta = \sum_{n \geq 1} n |\varphi_n|^2 + Q\varphi_0$$

The calculation of the conformal anomaly, which governs the background dependence, uses the same strategy as in the case of Liouville, except that instead of dealing with the usual bulk scalar field governed by the usual Laplace operator, we have to deal with the boundary Liouville field governed by the Dirichlet-to-Neumann operator.

The conformal anomaly is governed by the heat kernel expansion

$$\operatorname{tr}(f e^{-t\mathscr{O}}) = \frac{a_0[f]}{t} + a_1[f] + O(t \ln t)$$

$$a_0[f] = \frac{1}{\pi} \oint f \; ds \quad \text{(Weyl's law)}$$

$$a_1[f] = a_1 \oint k \, \mathrm{d}s \quad \text{(general form)}$$

Crucial result (J. Edward and S. Wu 1991) $a_1 = 0$

$$\frac{\det' \mathscr{O}}{\ell} \text{ is a conformal invariant}$$

(Colin Guillarmou and Laurent Guillopé 2007)

Upshot: $D_{g_0}\sigma \, e^{\frac{c-24}{24\pi}\oint \sigma \mathscr{O} \sigma}$

yields a background-independent theory.

$$D\varphi\, e^{-\mathcal{S}[\varphi]} = \mathrm{d}\varphi_0 \prod_{n\geq 1} (\mathrm{d}\,\mathrm{Re}\,\varphi_n \mathrm{d}\,\mathrm{Im}\,\varphi_n)\, e^{-\mathcal{S}[\varphi]}$$

$$\mathcal{S} = S_0 + S_\Lambda$$

$$S_0 = \frac{1}{4\pi}\int_0^{2\pi} \Big(\varphi\partial_\rho\phi(\theta,\rho=1) + 2Q\varphi\Big)\mathrm{d}\theta = \sum_{n\geq 1} n|\varphi_n|^2 + Q\varphi_0$$

$$S_\Lambda = \frac{\Lambda}{16\pi}\int_{\mathscr{D}} e^{2\phi/Q}\,\mathrm{d}^2 z$$

The action is PSL(2,R)-invariant, and predicts $\varphi_n \sim \frac{1}{\sqrt{n}}$

almost surely, implying that the boundary field is distribution-valued (very unlike a normal one-dimensional quantum variable q, which would have modes going like 1/n and would be continuous).

JT gravity on finite geometries: CFT

We don't have time to go through the derivation today, but the result is extremely simple and elegant.

$$\phi = \psi + \chi$$

ψ is a Coulomb gas field with Neumann boundary conditions. χ is a timelike direction with Dirichlet boundary conditions (so phi is like a lightlike direction). In the bulk the quantum fluctuations of ψ and χ cancels.

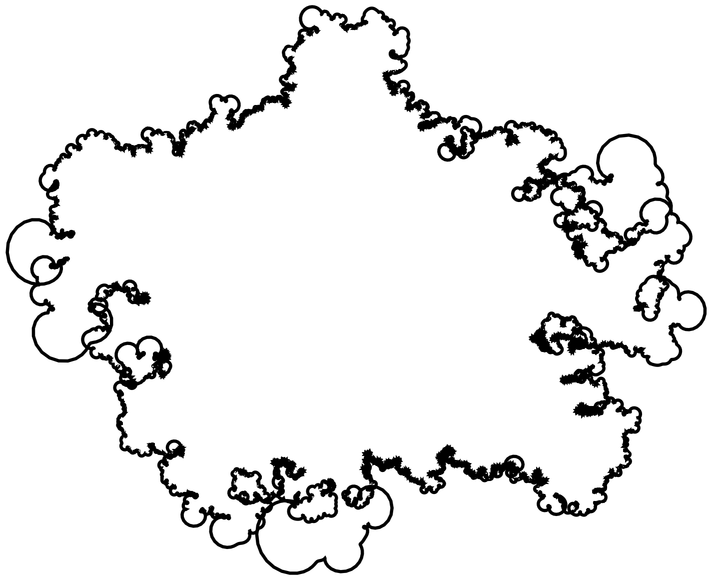
$$S = \frac{1}{4\pi}\left[\int (g_0^{\mu\nu}\partial_\mu\psi\partial_\nu\psi + QR_0\psi)\mathrm{d}_{g_0}x + 2Q\oint k_0\psi \mathrm{d}s_0\right] - \frac{1}{4\pi}\int g_0^{\mu\nu}\partial_\mu\chi\partial_\nu\chi$$
$$+\frac{\eta Q}{2\pi L^2}\int \chi e^{2(\psi+\chi)/Q}\mathrm{d}_{g_0}x + \frac{\Lambda}{16\pi}\int e^{2(\psi+\chi)/Q}\mathrm{d}_{g_0}x$$

$$\beta = \int_0^{2\pi} :e^{\frac{\gamma}{2}\varphi}: \mathrm{d}\theta$$

Going from finite-size to infinite geometry

Can we find the Schwarzian description as a limit of the finite-size geometry description?

$$\Lambda \to -\infty \qquad\qquad \beta_{\mathrm{S}} \sim \frac{\beta}{L^2}(|\Lambda|L)^{d_{\mathrm{H}}-2}$$



Idea: on distance scales much larger than the curvature length scales, the boundary looks smooth. The "smoothly wiggling" behaviour could emerge as a hydrodynamic, long distance approximation.

There is a very natural candidate for the emerging reparameterization mode:

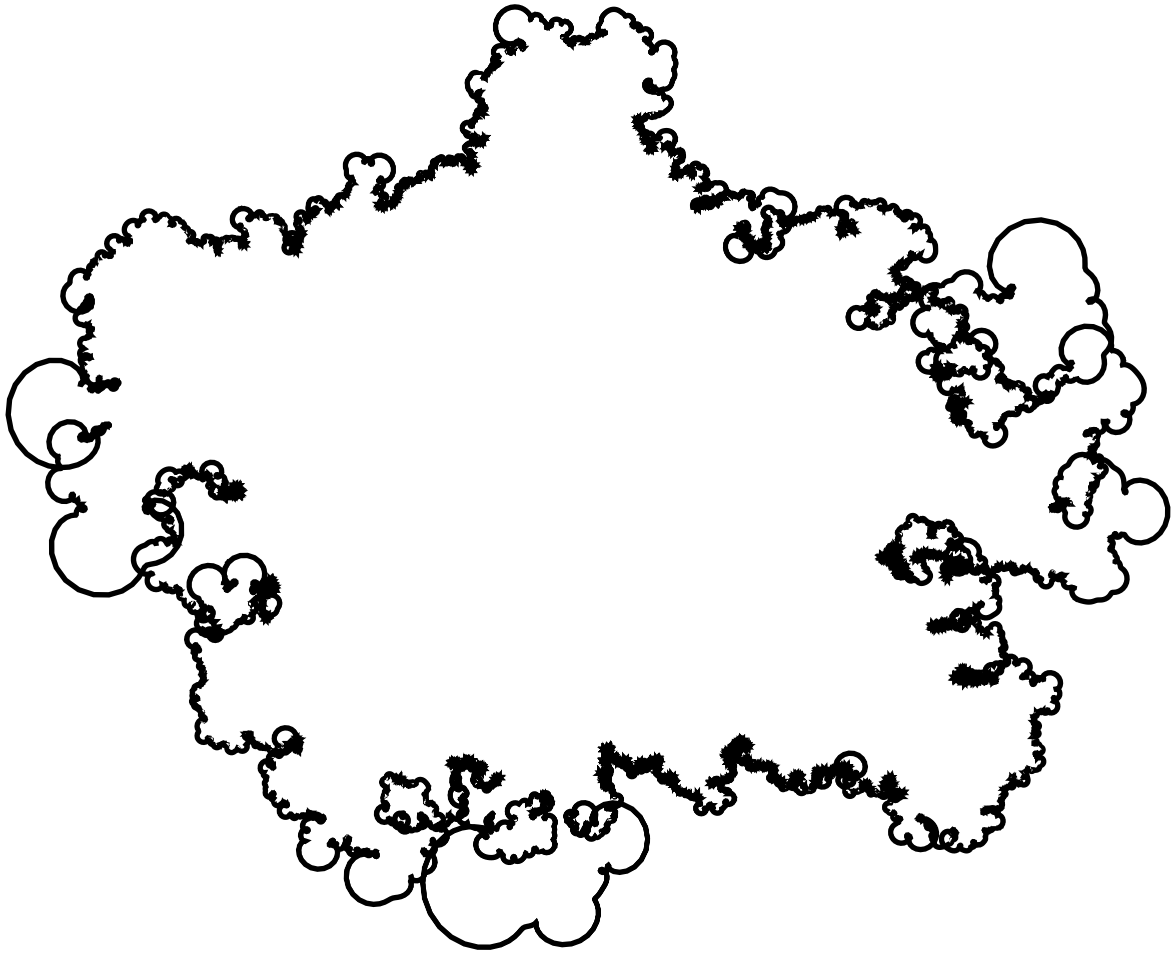
$$f(\theta) = \frac{2\pi}{\beta} \int_0^\theta : e^{\frac{\gamma}{2}\varphi(\theta')} : \, \mathrm{d}\theta'$$

Conjecture: the law of f is given by the Schwarzian measure in the limit

$$\Lambda \to -\infty \qquad\qquad \beta_{\mathrm{S}} \sim \frac{\beta}{L^2}(|\Lambda| L)^{d_{\mathrm{H}}-2}$$

Conceptually, this limit is highly non-trivial: time emerges.

Very generally, one can conjecture that the boundary Hamiltonian is an emergent concept, providing a hydrodynamic description of the gravity fluctuations on very large distance scales. This is probably what TTbar describes.



Thank you for your attention!