Correlation functions in 2D critical loop models

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Probabilistic paths to QFT, 27/04/2026

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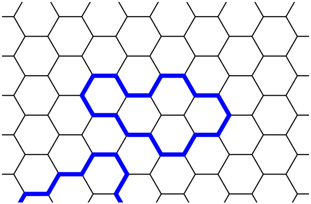
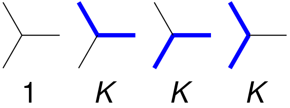
Loops: from lattice model to continuum limit

loop weight n



Vertex weights 1 and K



$$K_c = \left(2 \pm \sqrt{2-n}\right)^{-1/2}$$

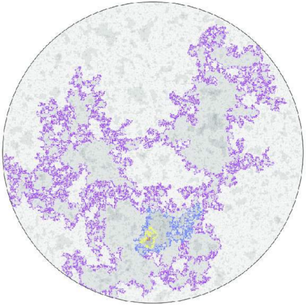


Figure from Miller-Watson-Wilson ’16

(colours show level of nesting)

- Loop model with $|n| \leq 2$
- CFT with central charge c
- CLE_κ with parameter κ

The task

Compute correlation functions

Motivation from statistical physics [Nienhuis, Baxter, Cardy,...]

$$Z(K,n) = \sum_{\text{loops}} K^{\#\text{monomers}} n^{\#\text{loops}}, \qquad K_c = \left(2 \pm \sqrt{2-n}\right)^{-1/2}$$

where $-2 \leq n \leq 2$. Plus (minus) sign for the dilute (dense) phase.

Special cases

- $n = 1$ dense: Site percolation
- $n = 1$ dilute: Ising model
- $n \to 0$ dilute: Self-avoiding walks
- $n = 2$ either: Gaussian free field, XY model
- $Q \to 0$ Potts: Uniform spanning trees (or forests via correlators)

All of these are really *logarithmic* CFTs.

Our first objective is to understand the case of 'generic' n, where logarithms play only a subdued role.

CFT of the $O(n)$ model [Di Francesco-Saleur-Zuber '87]

Central charge from loop weight $n = -2\cos(\pi\beta^2)$:

$$c = 13 - 6\beta^2 - 6\beta^{-2} \qquad \text{with} \qquad \begin{cases} \Re\beta^2 > 0 \,, \\ \beta^2 \notin \mathbb{Q} \,. \end{cases}$$

Conformal weight Δ and momentum P:

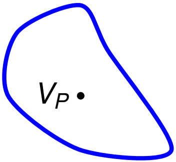
$$\Delta = P^2 - P^2_{(1,1)} \,, \quad \Delta_{(r,s)} = P^2_{(r,s)} - P^2_{(1,1)} \,, \quad P_{(r,s)} = \frac{1}{2}\left(-\beta r + \beta^{-1} s\right) \,.$$

Field content, with left- and right-moving conformal weights $(\Delta, \bar{\Delta})$:

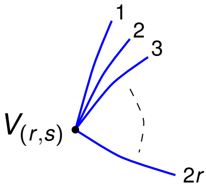
Name	Notation	Parameters	$(\Delta, \bar{\Delta})$
Degenerate	$V^d_{\langle r,s\rangle}$	$r = 1; s \in 2\mathbb{N}+1$	$\left(\Delta_{(r,s)}, \Delta_{(r,s)}\right)$
Diagonal	V_P	$P \in \mathbb{C}$	$\left(P^2 - P^2_{(1,1)}, P^2 - P^2_{(1,1)}\right)$
Non-diagonal	$V_{(r,s)}$	$r \in \frac{1}{2}\mathbb{N}^*; s \in \frac{1}{r}\mathbb{Z}$	$\left(\Delta_{(r,s)}, \Delta_{(-r,s)}\right)$

Interpretation of fields within the loop model

- Degenerate $V^d_{\langle 1,1\rangle}$: Identity operator.
- Degenerate $V^d_{\langle 1,3\rangle}$: Energy operator (measures monomer density).
- Diagonal V_P: Changes weight of surrounding loop from n to $w(P)$.
- Non-diagonal field $V_{(r,s)}$: Inserts $2r$ segments with momentum s.



$$w(P) = 2\cos(2\pi\beta P)$$

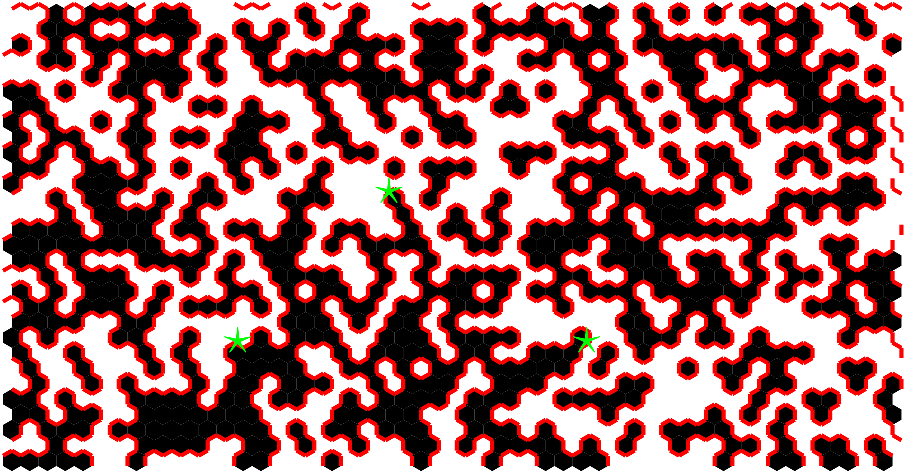


Phase $\exp\left(\frac{i}{2}s\sum_{k=1}^{2r}\theta_k\right)$

- $\#\text{edges} = \sum_{i=1}^{n} r_i \in \mathbb{N}$ in correlation functions.
- Spin $rs \in \mathbb{Z}$ of $V_{(r,s)}$ for singlevaluedness.

Correlation functions

For example: Site percolation ($n = 1$, dense phase)



$P(z_1, z_2, z_3 \in \text{same loop}) = \frac{C_{(1,0)(1,0)(1,0)}}{|z_1-z_2|^{2\Delta}|z_2-z_3|^{2\Delta}|z_3-z_1|^{2\Delta}}$ by global conf. inv.

$\Delta = \Delta_{(1,0)} = \frac{1}{8}$ and $C_{(1,0)(1,0)(1,0)}$ given below [Ang–Cai–Sun–Wu '24, JNRiRo 2510.04701]

Torus partition function [Di Francesco-Saleur-Zuber '87]

Spectrum and operators from modular invariant torus partition function:

$$Z^{O(n)}(q) = \sum_{s\in 2\mathbb{N}+1} \chi_{\langle 1,s\rangle}(q) + \sum_{r\in\frac12\mathbb{N}^*} \sum_{s\in\frac1r\mathbb{Z}} L_{(r,s)}(n)\chi^N_{(r,s)}(q)\,, \qquad q = e^{2\pi i\tau}$$

with diagonal degenerate and non-diagonal characters

$$\chi_{\langle r,s\rangle}(q) = \left|\frac{q^{P^2_{(r,s)}} - q^{P^2_{(r,-s)}}}{\eta(q)}\right|^2 \,, \qquad \chi^N_{(r,s)}(q) = \frac{q^{P^2_{(r,s)}}\bar{q}^{P^2_{(r,-s)}}}{|\eta(q)|^2}\,.$$

Field $V_{(r,s)}$ has rep $\mathscr{W}_{(r,s)}$ with character $\chi^N_{(r,s)}(q)$ if $r\neq\mathbb{Z}^*$ or $s\neq\mathbb{Z}^*$, but indecomposable with character $\chi^N_{(r,s)}(q)+\chi^N_{(r,-s)}(q)$ if $r,s\in\mathbb{N}^*$. [Gorbenko-Zan '20]

Polynomial multiplicities (can also be written as Ramanujan sums)

$$L_{(r,s)}(n) = \delta_{r,1}\delta_{s\in 2\mathbb{Z}+1} + \frac{1}{2r}\sum_{r'=0}^{2r-1} e^{\pi i r' s} x_{(2r)\wedge r'}(n)\,,$$

$$x_0(n) = 2 \quad , \quad x_1(n) = n \quad , \quad n x_d(n) = x_{d-1}(n) + x_{d+1}(n)\,.$$

Global $O(n)$ symmetry [GNJRiS 2111.01106]

$L_{(r,s)}(n)$ reveal a bimodule structure of the $O(n)$ CFT's state space

$$\mathcal{S}^{O(n)} = \bigoplus_{s\in 2\mathbb{N}+1} [\,] \otimes \mathscr{R}_{\langle 1,s\rangle} \oplus \bigoplus_{r\in\frac12\mathbb{N}^*} \bigoplus_{s\in\frac1r\mathbb{Z}} \Lambda_{(r,s)} \otimes \mathscr{W}_{(r,s)}$$

where $\Lambda_{(r,s)}$ decompose over $O(n)$ irreps such that [Binder-Rychkov '19]

$$\dim_{O(n)} \Lambda_{(r,s)} = L_{(r,s)}(n)\,.$$

Examples of this decomposition:

$$\Lambda_{(\frac12,0)} = [1]\,,\qquad \Lambda_{(1,0)} = [2]\,,\qquad \Lambda_{(1,1)} = [11]\,,\qquad \Lambda_{(\frac32,0)} = [3]+[111]\,,$$

$$\Lambda_{(\frac32,\frac23)} = [21]\,,\qquad \Lambda_{(2,0)} = [4]+[22]+[211]+[2]+[\,]\,,\qquad \Lambda_{(2,\frac12)} = [31]+[211]+[11]\,,\qquad \Lambda_{(2,1)} = [31]+[22]+[1111]+[2]\,.$$

Interchiral conformal bootstrap

Consider first the sphere geometry.

- Two-point functions are given by the conformal dimensions, up to normalisation of the field.
- Three-point functions are also fixed by global conformal invariance, up to structure constants (see below).
- Four-point functions containing a degenerate operator obey a BPZ differential equation: not our case of interest.
- $V^d_{\langle 1,s\rangle}$ gives helpful shift equation (see below).
- Minimal models and Liouville have also $V^d_{\langle r,1\rangle}$, implying solvability.
- Not the case here: therefore we need the *conformal bootstrap*.

But we can do better than usual for two reasons:

- $V^d_{\langle 1,3\rangle}$ generates an *interchiral symmetry*. [HJS 2005.07258]
- The global $O(n)$ symmetry is helpful. [GNJRiS 2111.01106]
- But symmetry is larger than $O(n)$ and turns out crucial. [JRoS '26]

- Consider a four-point function of non-diagonal primary fields, and its s-channel decomposition into conformal blocks:

$$\left\langle \prod_{i=1}^{4} V_{(r_i,s_i)} \right\rangle = \sum_{s\in 2\mathbb{N}+1} D_s \mathcal{G}^D_{\langle 1,s\rangle} + \sum_{r\in\frac12\mathbb{N}^*} \sum_{s\in\frac1r\mathbb{Z}} D_{(r,s)} \mathcal{G}_{(r,s)} \ .$$

- The blocks are known from Zamolodchikov's recursion relation.
- Degenerate shift eqs using $V^d_{\langle 1,3\rangle}$ determine $\frac{D_{(r,s+1)}}{D_{(r,s-1)}}$ and $\frac{D_{s+1}}{D_{s-1}}$.
- Rewrite in terms of interchiral blocks:

$$\mathcal{H}_{s_0} = \sum_{s\in s_0+2\mathbb{N}} \frac{D_s}{D_{s_0}} \mathcal{G}^D_{\langle 1,s\rangle} \quad , \quad \mathcal{H}_{(r,s)} = \sum_{j\in 2\mathbb{N}} \frac{D_{(r,s+j)}}{D_{(r,s)}} \mathcal{G}_{(r,s+j)} \ .$$

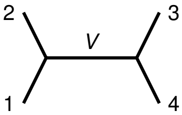
$$\left\langle \prod_{i=1}^{4} V_{(r_i,s_i)} \right\rangle = D_{s_0}\mathcal{H}_{s_0} + \sum_{r\in\frac12\mathbb{N}^*} \sum_{s\in\frac1r\mathbb{Z}\cap(-1,1]} D_{(r,s)} \mathcal{H}_{(r,s)} \ ,$$

- Correponding interchiral representations

$$\widetilde{\mathcal{R}}_{\langle 1,1\rangle} = \bigoplus_{s\in 2\mathbb{N}+1} \mathcal{R}_{\langle 1,s\rangle} \ , \qquad \widetilde{\mathcal{W}}_{(r,s)} = \bigoplus_{s'\in 2\mathbb{Z}+s} \mathcal{W}_{(r,s')} \ .$$

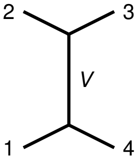
Solve then the crossing equations

$$\sum_{V\in\mathcal{S}^{(s)}} D_V^{(s)}$$



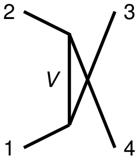
s-channel

$$= \sum_{V\in\mathcal{S}^{(t)}} D_V^{(t)}$$



t-channel

$$= \sum_{V\in\mathcal{S}^{(u)}} D_V^{(u)}$$



u-channel

We know the spectrum. Helpful to study the solutions in view of the global $O(n)$ symmetry. The solution space can be constrained by a characteristic ("signature") of a combinatorial map (see below).

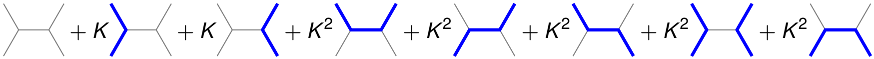
In favourable cases this gives a unique (numerical) solution.

Conjecture: Each solutions to the crossing equations gives a valid correlation function in the $O(n)$ CFT.

We have computed the 30 correlation functions with $\sum_{i=1}^{4} r_i = 2, 3, 4$.

Relation with diagram algebras [JRiS 2208.14298]

All configurations can be built by a transfer matrix (local piece shown):

$\check{R}_k =$ 

- $T = \left(\prod_{k=1}^{L} \check{R}_{2k}\right)\left(\prod_{k=1}^{L} \check{R}_{2k-1}\right)$ belongs to the unoriented Jones-Temperley-Lieb algebra $u\mathscr{JTL}_L(n)$.
- Acts on standard modules $W^{(L)}_{(r,s)}$ with $2r$ defects and momentum s.
- On $\mathcal{S}_L^{O(n)} = [1]^{\otimes L}$, commutant of $O(n)$ is the Brauer algebra $\mathscr{B}_L(n)$.
- From the branching rules $\mathscr{B}_L(n) \downarrow u\mathscr{JTL}_L(n)$ we find

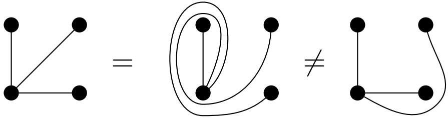
$$\mathcal{S}_L^{O(n)} \underset{u\mathscr{JTL}_L(n)\times O(n)}{=} \bigoplus_{r=\left\{\frac{L}{2}\right\}}^{\frac{L}{2}} \bigoplus_{\substack{s\in\frac{1}{r}\mathbb{Z}\\ -1<s\leq 1}} W^{(L)}_{(r,s)} \otimes \Lambda_{(r,s)} .$$

- Conjecture: $\lim_{\text{critical}} u\mathscr{JTL}_L(n) = \widetilde{\mathfrak{C}}_{\beta^2}$ and $\lim_{\text{critical}} W^{(L)}_{(r,s)} = \widetilde{\mathscr{W}}_{(r,s)}$.

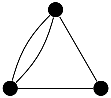
Combinatorial maps [GJNRiS 2302.08168]

A (connected) *combinatorial map* is a (connected) graph, together with a cyclic permutation of the half-edges around each of the N vertices. Monogons are forbidden.

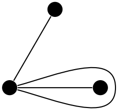
- $N \geq 4$: several cmaps for the same choice of non-diagonal fields:



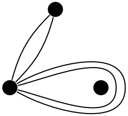
- $N = 3$: For each choice of fields the cmap is unique.
- If one field is diagonal, at least one pair of legs from a non-diagonal field must enclose it.



3, 3, 2 legs



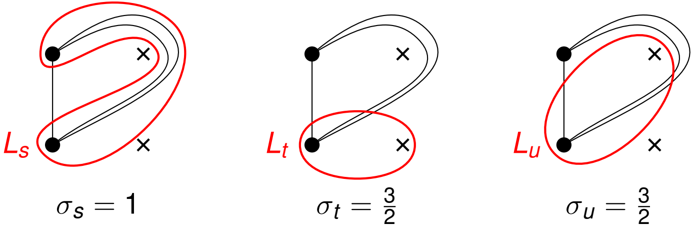
4, 1, 1 legs



6, 2, 0 legs

$N = 4$ cmaps and bootstrap on the sphere

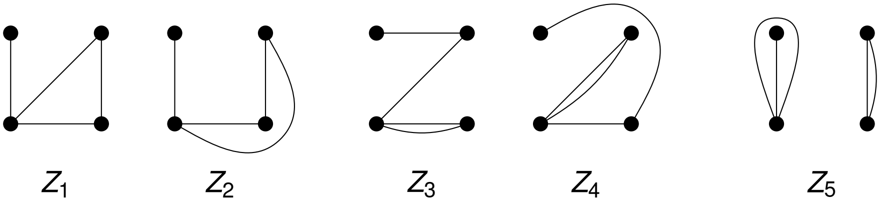
- For each channel $x = s, t, u$ define the *signature* σ_x as $\frac{1}{2}\times$ the least number of edges cut by a splitting contour:



- This constrains the spectra $\mathcal{S}^{(x)}$ of exchanged fields: $r \geq \sigma_x$.
- Conjecture: Dimension of bootstrap solution space = # cmaps.
- Signatures help to isolate the correlator defined by a given cmap.

Example: $\left\langle V_{(\frac{3}{2},0)} V_{(\frac{1}{2},0)} V_{(1,0)} V_{(1,0)} \right\rangle$

- Bootstrap equations have 5-dimensional solution space.
- We have indeed 5 cmaps:



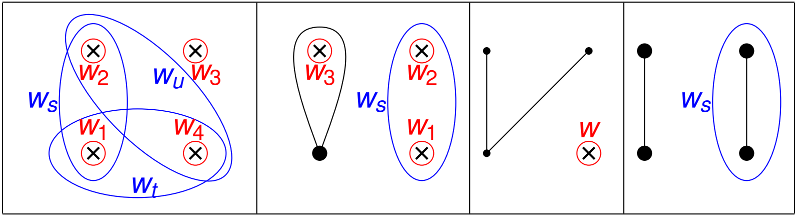
But the first two cmaps are not characterised by their signature:

Map	Z_1	Z_2	Z_3	Z_4	Z_5
Signature	$(1, \frac{3}{2}, \frac{3}{2})$	$(1, \frac{3}{2}, \frac{3}{2})$	$(2, \frac{1}{2}, \frac{3}{2})$	$(2, \frac{3}{2}, \frac{1}{2})$	$(0, \frac{5}{2}, \frac{5}{2})$

So extra considerations are needed to obtain Z_1 and Z_2 individually.

Factorisation and magic in 4-pt functions [NRiJ 2311.17558]

- Loops may surround field insertions i and channel contours x.
- Weights w_i ($i = 1, 2, 3, 4$) and w_x ($x = s, t, u$) in addition to n.



- Transfer matrix, for cylinder of circumference L.
- Isolate amplitude of exchanged field ω in the s-channel:

$$C^{\text{loop}}(L, \ell | K, n, w_i, w_x) = \sum_{\omega \in S(L)} A_\omega(L|K, n, w_i, w_x) \left(\frac{\Lambda_\omega(L|K, n, w_s)}{\Lambda_{\max}(L|K, n, w_s)} \right)^\ell$$

Key observation [HGJS 2002.09071]

$$\frac{A_{(r,s),\rho}(L|K,n,w_i,w_x)}{A_{(r,s),\rho}(L|K,n,w_i,w'_x)} = \frac{D^{(s)}_{(r,s)}(n,w_i,w_x)}{D^{(s)}_{(r,s)}(n,w_i,w'_x)}$$

Amplitude ratios are constant on the modules (r,s), independent of size L, independent of criticality K, and equal to the CFT ratios.

$$\frac{D^{(x)}_{(r,s)}}{D^{(x)\text{ref}}_{(r,s)}} = d^{(x)}_{(r,s)} + \delta_{\sigma^{(x)},0}\delta_{s\in\mathbb{Z}} \frac{f^{(x)}_{(r,s)}}{w_x - w(P_{(r,s)})}$$

- $d^{(x)}_{(r,s)}$ depend polynomially on all loop weights: n, w_i and w_x.
  The dependence on w_x becomes polynomial after we subtract a rational term that is needed for the 4-point function to be holomorphic in P_x.
- Found case-by-case from numerical bootstrap [NRiJ 2311.17558]
- Computed systematically from lattice algebras [JRoS ’26]

Digression on the Barnes double gamma function

Recall $c = 13 - 6\beta^2 - 6\beta^{-2}$ and set $Q = \beta + \beta^{-1}$.

For $\Re x > 0$ define $\Gamma_\beta(x)$ through

$$\log \Gamma_\beta(x) = \int_0^\infty \frac{\mathrm{d}t}{t}\left[\frac{\mathrm{e}^{-xt} - \mathrm{e}^{-Qt/2}}{(1-\mathrm{e}^{-\beta t})(1-\mathrm{e}^{-t/\beta})} - \frac{(Q/2-x)^2}{2\mathrm{e}^t} - \frac{Q/2-x}{t}\right]$$

and the shift equations

$$\frac{\Gamma_\beta(x+\beta)}{\Gamma_\beta(x)} = \sqrt{2\pi}\frac{\beta^{\beta x - \frac{1}{2}}}{\Gamma(\beta x)} \quad , \quad \frac{\Gamma_\beta(x+\beta^{-1})}{\Gamma_\beta(x)} = \sqrt{2\pi}\frac{\beta^{\frac{1}{2}-\beta^{-1}x}}{\Gamma(\beta^{-1}x)} .$$

Magic redux: $\frac{D^{(x)}_{(r,s)}}{D^{(x)\text{ref}}_{(r,s)}} = d^{(x)}_{(r,s)} + \delta_{\sigma(x),0}\delta_{s\in\mathbb{Z}}\frac{f^{(x)}_{(r,s)}}{w_x - w(P_{(r,s)})}$

Reference 4-point struct const is what we would get if things factorised

$$D^{(x)\text{ref}}_{(r,s)} = C^{\text{ref}}_{(r_1,s_1)(r_2,s_2)(r,s)} C^{\text{ref}}_{(r,s)(r_3,s_3)(r_4,s_4)} / B^{\text{ref}}_{(r,s)}$$

into 2- and 3-point structure constants:

$$C^{\text{ref}}_{(r_1,s_1)(r_2,s_2)(r_3,s_3)} = \prod_{\epsilon_1,\epsilon_2,\epsilon_3=\pm} \Gamma_\beta^{-1}\left(\tfrac{\beta+\beta^{-1}}{2} + \tfrac{\beta}{2}\left|\textstyle\sum_i \epsilon_i r_i\right| + \tfrac{\beta^{-1}}{2}\textstyle\sum_i \epsilon_i s_i\right)$$

The residues are known, so the only outstanding issue is $d^{(x)}_{(r,s)}$:

$$f^{(x)}_{(r,s)} = \frac{\underset{w=w(P_{(r,s)})}{\text{Res}} D_{(r,s)}(w)}{D^{\text{ref}}_{(r,s)}} \underset{r\in\mathbb{N}^*,s\in\mathbb{N}}{=} -(-)^{(r+1)s}\rho^{r,s}_{(r_1,s_1)(r_2,s_2)}\rho^{r,s}_{(r_4,s_4)(r_3,s_3)}$$

$$\rho^{r,s}_{(r_1,s_1)(r_2,s_2)} = (-)^{s\min(r,|r_1-r_2|)\delta_{r_1<r_2}} \prod_{\pm} \prod_{j\overset{1}{=}-\frac{r-1-|r_1\pm r_2|}{2}}^{\frac{r-1-|r_1\pm r_2|}{2}} 2\cos\pi\left(j\beta^2 + \frac{s-s_1\mp s_2}{2}\right)$$

Three-point functions on the sphere [JNRiRo 2510.04701]

$$C^{\text{ref}}_{(r_1,s_1)(r_2,s_2)(r_3,s_3)} = \prod_{\epsilon_1,\epsilon_2,\epsilon_3=\pm} \Gamma_\beta^{-1}\left(\frac{\beta+\beta^{-1}}{2} + \frac{\beta}{2}\left|\sum_i \epsilon_i r_i\right| + \frac{\beta^{-1}}{2}\sum_i \epsilon_i s_i\right) .$$

For diag fields, set $V_P = V_{(0,2\beta P)}$, so $C^{\text{ref}}_{(0,2\beta P_1)(0,2\beta P_2)(0,2\beta P_3)} = C_{P_1,P_2,P_3}$.

We observe that (up to normalisation by two-point functions):

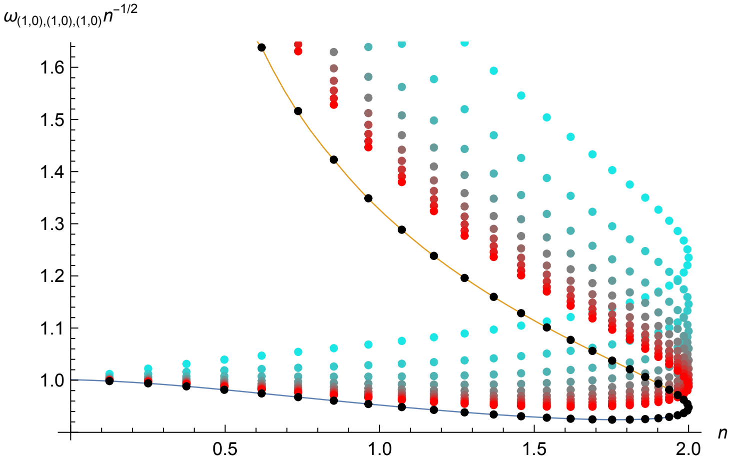
- C_{P_1,P_2,P_3} equals the structure constants of diagonal fields [Ikhlef-JJ-Saleur '16, Ang–Cai–Sun–Wu '24]. (= imaginary DOZZ formula)
- $C^{\text{ref}}_{(1,0)(1,0)(1,0)}$ related with $P(z_1, z_2, z_3 \in \text{same loop})$ [Ang–Cai–Sun–Wu '24].

Conjecture for three-point structure constants of cmaps

Given by $C^{\text{ref}}_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}$, by taking a ratio that is invariant under field renormalizations $V_{(r,s)} \to \lambda_{(r,s)} V_{(r,s)}$:

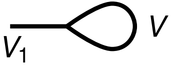
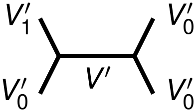
$$\omega_{123} = C^{\text{ref}}_{123}\sqrt{\frac{C^{\text{ref}}_{000}}{C^{\text{ref}}_{011}C^{\text{ref}}_{022}C^{\text{ref}}_{033}}} , \qquad 0 = \mathrm{Id} = V_{(0,1-\beta^2)} .$$

Numerical check for the case of 3 points $\in$ same loop:



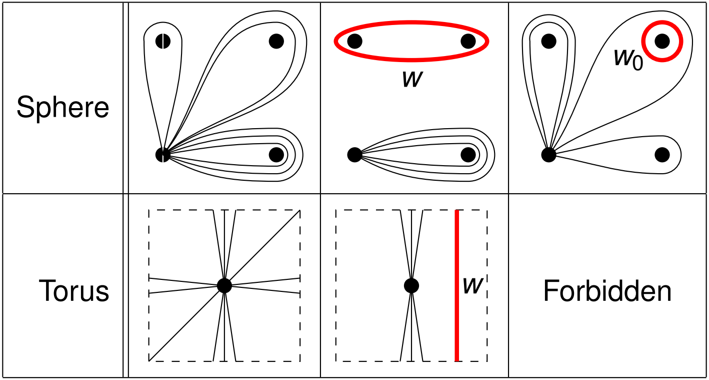
This case is the one proved by [Ang–Cai–Sun–Wu '24].

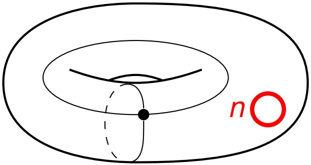
One-point functions on the torus [RoRiJ 2604.24491]

		Torus	Sphere
Correlation function		$\left\langle V_1 \right\rangle$	$\left\langle V'_0 V'_1 V'_0 V'_0 \right\rangle$
Conformal block			
Central charge parameter		β	$\frac{\beta}{\sqrt{2}}$
Loop weight		n	$-\sqrt{2-n}$
External field	Momentum	P_1	$\frac{P_1}{\sqrt{2}}$
	Kac indices	(r_1, s_1)	$(r_1, \frac{s_1}{2})$
	Weight	w_1	$\sqrt{w_1+2}$
	Spin	S_1	$\frac{S_1}{2}$
Channel field	Momentum	P	$\sqrt{2}P$
	Kac indices	(r, s)	$(2r, s)$
	Weight	w	w
	Spin	S	$2S$

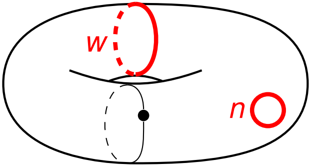
The sphere–torus correspondence

Think of the sphere as a "pillow" double-cover of the torus with V_0' at corners 1, τ, $1+\tau$ and $V_1 = V_{(r_1,s_1)}$ at 0. Take $V_0' = V_{(0,\frac{1}{2})}$ so $w_0 = 0$:

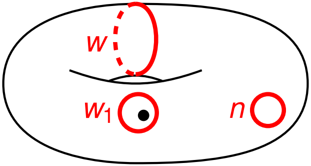




Map $(1, 1, 0)$



Map $(1, 0, 0)$



Map $(0, 0, 0)$

Resulting decomposition:

$$\left\langle V_{(r_1,s_1)} \right\rangle = \sum_{(r,s)\in\mathcal{S}} D_{(r,s)} \mathscr{G}_{(r,s)} , \qquad \mathscr{G}_{(r,s)} = \mathscr{F}_{\Delta_{(r,s)}} \mathscr{F}_{\Delta_{(r,-s)}}$$

with structure reminiscent of the sphere case

$$\frac{D_{(r,s)}}{D^{\text{ref}}_{(r,s)}} = \frac{1}{\kappa_{(r,s)}} \left(\frac{d_{(r,s)}}{r} + \delta_{r\in\mathbb{N}^*} \delta_{s\in\mathbb{Z}} \frac{\theta_1^{r,s}}{w - w_{(r,s)}} \right)$$

- Bootstrapped all $\left\langle V_{(r_1,s_1)} \right\rangle$ with $r_1 \leq 3$ and determined $d_{(r,s)}$.
- $\left\langle V_{P_{(1,1)}} \right\rangle$ recovers multiplicities $L_{(r,s)}(w)$ of torus partition function.

$D_{(r,s)} / D^{ref}_{(r,s)}$ can be found analytically [JRoS '26]

Comes from the symmetry of the diagram algebra, not from the CFT.

Compute a modified Markov trace of the Jones-Wenzl projector on $W^{(L)}_{(r,s)}$ in $u\mathscr{JTL}_L(n)$, keeping only terms corresponding to the chosen cmap. Result does not depend on L (if only $2r \leq L$).

Boundary conditions in loop models

Diagonal boundary conditions: preserve $O(n)$ symmetry

For $S \in \mathbb{N}^*$, project S strands on spin $S/2$ at boundary [JS '26]:

- $S = 1$: Free bcs.
- $S = 2$: Generalises "new" bcs in 3-state Potts [Affleck-Saleur '26]

They couple only to diagonal fields.

Non-diagonal boundary conditions: break $O(n)$ to a subgroup

- Modify weight of a boundary-touching loop from n to any $n_1 \in \mathbb{C}$ [JS math-ph/0611078]
- Dual of diagonal: $S = 1$ fixed/wired, $S = 2$ mixed,... [JS '26]
- Dirichlet: open strands can end at boundary [Dubail-J-S 0905:1382]

They couple also to non-diagonal fields.

- Focus mainly on diagonal boundary conds in the *disc geometry*.

Two well-known results

Crossing probability [Cardy '92, Smirnov '01]

- Probability that a percolation cluster connects two disjoint boundary arcs.
- Four-point function of boundary fields.

Left-passage probability [Schramm '01]

- Probability that a chordal SLE_κ trace goes to the left of a bulk point.
- One bulk field, two boundary fields.

- In both cases, a degenerate boundary field gives an ODE.
- Solution is a hypergeometric function: one conformal block.
- We study 1-point and 2-point functions of bulk fields. No ODE.
- The 2-point case gives an infinite sum over conformal blocks.

Two-point functions on UHP [DJNRiS 2512.10400, 2602.12000]

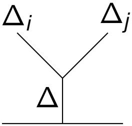
- Conformal invariance fixes two-point function on upper-half plane:

$$\langle V_i(z_1) V_j(z_2)\rangle = \frac{G_{ij}(\sigma)}{|z_1-\bar{z}_2|^{4\Delta_i}(z_2-\bar{z}_2)^{2\Delta_j-2\Delta_i}}$$

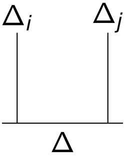
up to a function of the cross-ratio $\sigma = \frac{|z_1-z_2|^2}{(z_1-\bar{z}_2)(\bar{z}_1-z_2)}$. [Cardy-Lewellen '91]

- Bulk-boundary bootstrap eqs [Cardy-Lewellen '91, Lewellen '92]

$$\sum_{\Delta\in\mathcal{S}^{(s)}} D^{\text{bulk}}_{\Delta}$$



$$= \sum_{\Delta\in\mathcal{S}^{(t)}} D^{\text{bdy}}_{\Delta}$$



- We have $\langle V\rangle \neq 0$ only for diagonal or degenerate fields.
- Boundary spectrum is $\mathcal{S}^{(t)} = \left\{\psi^d_{\langle r,s\rangle}\right\}_{\substack{r\in\mathbb{N}^*\\ s=1,3,\dots,2S-1}}$
- In particular for $S=1$ (free or wired):
  $\mathcal{S}^{(s)} = \{\Delta_{(1,1+\mathbb{N})}\}$ and $\mathcal{S}^{(t)} = \{\Delta_{(1+\mathbb{N},1)}\}$

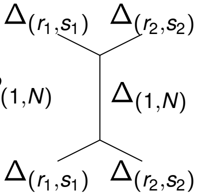
Two-point connectivity $\langle V_{(0,1/2)} V_{(0,1/2)} \rangle$

$P(z_1, z_2 \in \text{same FK cluster})$ with free $(+)$ or wired $(-)$ boundary cond:

$$G^{\text{free (wired)}}_{(0,\frac{1}{2})(0,\frac{1}{2})}(\sigma) = F^{(1)}_{(0,\frac{1}{2})(0,\frac{1}{2})}(\sigma) \pm F^{(2)}_{(0,\frac{1}{2},)(0,\frac{1}{2})}(\sigma)$$

with

$$F^{(k)}_{(r_1,s_1)(r_2,s_2)}(\sigma) = \sum_{N \overset{2}{=} k}^{\infty} C_{(r_1,s_1)(r_2,s_2)(1,N)} R_{(1,N)}$$

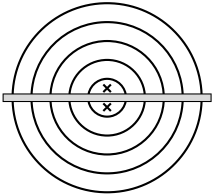


where $C_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}$ are 3-point structure constants as above, the diagram represents the conformal blocks, and

$$\langle V^d_{(1,N)}(z) \rangle = \frac{R_{(1,N)}}{|z - \bar{z}|^{2\Delta_{(1,N)}}}\,, \qquad R_{(1,N)} = \sin(2\pi\beta^{-1} P_{(1,N)})$$

Two diagonal operators $\langle V_{P_1} V_{P_2} \rangle$ [DJRiRo '26]

- On UHP, each loop gets weight w_1, w_2, w_{12} or n if it surrounds only z_1, only z_2, both or neither.
- Reference pieces can be worked out.
- Diagonal operators do not lead to polynomials in the bulk channel.
- But there is a family of polynomials in the boundary (t) channel. They are equal to this diagram in $\mathscr{TL}_L(n)$:



Rectangle $=$ Jones-Wenzl projector, and crosses represent z_1, z_2. It can be evaluated by a set of recursion relations.

Conclusions

We have or are close to analytical control of correlation functions of:

- Three and four points on the sphere (or plane).
- One point on the torus (and the annulus is under study).
- One and two points on the disc (or upper half plane).

Possible next steps:

- Operator product expansions in loop models.
- Non-diagonal boundary conditions.
- A Segal-like program that mixes bootstrap and lattice algebras.
- Applications to problems in physics.