The S-matrix bootstrap program in 1+1 dimensions: an overview

K. K. Kozlowski

CNRS, Laboratoire de Physique, ENS de Lyon.

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Outline

The overall setting

- 1+1 dimensional classical Sinh-Gordon evolution equation

$$\big(\partial_t^2 - \partial_x^2\big)\varphi(\boldsymbol{x}) + \frac{m^2}{g}\sinh[g\varphi(\boldsymbol{x})] = 0 \qquad \boldsymbol{x} = (t,x) \in \mathbb{R}^{1,1}\,, \quad \boldsymbol{x}^2 = t^2 - x^2$$

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- Quantisation of wave equations $\rightsquigarrow$ QFTs

$$\varphi(\boldsymbol{x}) \hookrightarrow \boldsymbol{\Phi} \,:\, \mathcal{S}(\mathbb{R}^{1,1}) \to \mathcal{L}(\mathfrak{h})\,, \qquad o_s(\boldsymbol{x}) \;\Big(e.g. \prod_{u=1}^{r}\big\{\partial_t^{\alpha_u}\partial_x^{\beta_u}\varphi(\boldsymbol{x})\big\}\cdot \mathrm{e}^{\gamma\varphi(\boldsymbol{x})}\Big) \hookrightarrow \mathcal{O}_s$$

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- Correlation functions $\equiv$ central objects

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- Wightman axioms ⇝ $\mathcal{W}(\boldsymbol{x}_1,\dots,\boldsymbol{x}_k)$
  - Poincaré invariance, Positivity, Hermiticity
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⊛ Verification of Osterwalder-Schrader Axioms $\Rightarrow$ QFT

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⊛ integrable path

- Existence of non-trivial S-matrices in 1+1 dimensions

Sinh-G : **’76** Gryanik, Vergeles

Extensions : **’77** Zamolodchikov, **’78** Karowski, Thun, **’79** Arinshtein, Fateyev, Z., **’79** $Z.^2$

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$$\left(\boldsymbol{\Omega}, \mathrm{e}^{\alpha \boldsymbol{\Phi}(t,x)} \mathrm{e}^{\alpha \boldsymbol{\Phi}(0)} \boldsymbol{\Omega}\right) \underset{\mathfrak{r} \to 0^+}{\sim} C_\alpha^{-2}\, \mathfrak{r}^{-4\alpha^2} \left(1 + \mathrm{o}(1)\right), \qquad \mathfrak{r} = \sqrt{x^2 - t^2}$$

$$C_\alpha = \exp\left\{ \int_0^{+\infty} \frac{\mathrm{d}t}{t} \left[\frac{\sinh^2(2\alpha t)}{2\sinh(gt)\sinh(g^{-1}t)\cosh[(g+g^{-1})t]} - 2\alpha^2 \mathrm{e}^{-\frac{2t}{g}} \right] \right\} \cdot \left\{ \frac{m}{4\sqrt{\pi}} \Gamma\left(\tfrac{1}{2(1+g^2)}\right) \Gamma\left(1 + \tfrac{1}{2(1+g^{-2})}\right) \right\}^{2\alpha^2}$$

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  - Explicit conjectures identifying effective CFTs & scaling dimensions

  '90 Zamolodchikov , **'90-'91** Klassen, Melzer , **'00** Korff

II) Bootstrap for the Sinh-Gordon QFT

Basics

'76 Gryanik, Vergeles Sinh-Gordon IQFT

$$\mathfrak{h} = \bigoplus_{n=0}^{+\infty} L^2(\mathbb{R}^n_>) \qquad \text{with} \qquad \mathbb{R}^n_> = \left\{ \boldsymbol{\beta}_n = (\beta_1, \dots, \beta_n) \in \mathbb{R}^n \;:\; \beta_1 > \cdots > \beta_n \right\}$$

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$$\mathrm{S}(\beta) = \frac{\tanh[\frac{1}{2}\beta - \mathrm{i}\pi\mathfrak{b}]}{\tanh[\frac{1}{2}\beta + \mathrm{i}\pi\mathfrak{b}]} \qquad \text{with} \qquad \mathfrak{b} = \frac{1}{2}\,\frac{g^2}{8\pi + g^2} \in \big[\, 0 \,;\, \tfrac{1}{2} \,\big]$$

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Translation operator by $\boldsymbol{y} = (y_0, y_1)$

$$\mathrm{U}_{\mathrm{T}_{\boldsymbol{y}}} \cdot \boldsymbol{f} = \Big(\mathrm{U}^{(0)}_{\mathrm{T}_{\boldsymbol{y}}} \cdot f^{(0)}, \dots, \mathrm{U}^{(n)}_{\mathrm{T}_{\boldsymbol{y}}} \cdot f^{(n)}, \dots \Big) \qquad \text{where} \qquad \mathrm{U}^{(n)}_{\mathrm{T}_{\boldsymbol{y}}} \cdot f^{(n)}(\boldsymbol{\beta}_n) = \prod_{a=1}^{n} \mathrm{e}^{\mathrm{i}\boldsymbol{p}(\beta_a)*\boldsymbol{y}} \cdot f^{(n)}(\boldsymbol{\beta}_n)$$

with $\quad \boldsymbol{p}(\beta) = \big(m\cosh(\beta), m\sinh(\beta)\big) \qquad$ and $\qquad \boldsymbol{x} * \boldsymbol{y} = x_0 y_0 - x_1 y_1$

The field operators

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- Space-time dependence

$$\mathcal{O}(\boldsymbol{x}) = \mathrm{U}_{\mathrm{T}_{\boldsymbol{x}}} \cdot \mathcal{O}(\boldsymbol{0}) \cdot \mathrm{U}_{\mathrm{T}_{\boldsymbol{x}}}^{-1}$$

The explicit construction

Construction of $\mathcal{O}_{nm}(\boldsymbol{x})$ $\rightsquigarrow$ Bootstrap program

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$$\big(\mathcal{O}(\boldsymbol{x})\cdot\boldsymbol{f}\big)^{(n)}(\alpha_n) = \int_{\mathbb{R}^m_>} \mathrm{d}^m\beta \prod_{a=1}^{m}\Big\{\mathrm{e}^{-\mathrm{i}\boldsymbol{p}(\beta_a)*\boldsymbol{x}}\Big\}\prod_{a=1}^{n}\Big\{\mathrm{e}^{\mathrm{i}\boldsymbol{p}(\alpha_a)*\boldsymbol{x}}\Big\}\mathcal{M}^{(\mathcal{O})}_{n;m}\big(\alpha_n;\boldsymbol{\beta}_m\big)\, f^{(m)}\big(\boldsymbol{\beta}_m\big)$$

$\mathcal{M}^{(\mathcal{O})}_{n;m}\big(\alpha_n;\boldsymbol{\beta}_m\big) \equiv$ generalised function

The explicit construction

Construction of $\mathcal{O}_{nm}(\boldsymbol{x})$ $\rightsquigarrow$ Bootstrap program

'78 Karowski, Weisz , **'84-'86** Smirnov , **'87-'89** Kirillov,Smirnov , **'88** Khamitov

- 0-particle component $\big(\mathcal{O}(\boldsymbol{x})\cdot \boldsymbol{f}\big)^{(0)} = \sum_{m\geq 0} \mathcal{O}_{0m}(\boldsymbol{x})\cdot f^{(m)}$

$$\mathcal{O}_{0m}(\boldsymbol{x})\cdot f^{(m)} = \int\limits_{\mathbb{R}^m_>} \mathrm{d}^m\beta\, \mathcal{F}^{(\mathcal{O})}_{m;+}(\boldsymbol{\beta}_m) \prod_{a=1}^{m}\Big\{\mathrm{e}^{-\mathrm{i}\boldsymbol{p}(\beta_a)*\boldsymbol{x}}\Big\}\, f^{(m)}(\boldsymbol{\beta}_m)$$

Form factor $\mathcal{F}^{(\mathcal{O})}_{m;+} \equiv +$ boundary value of $\mathcal{F}^{(\mathcal{O})}_{m} \in \mathcal{M}\Big(\big\{\boldsymbol{\beta}_m \ :\ 0 < \Im(\beta_a) < \pi\big\}\Big)$

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$\mathcal{M}^{(\mathcal{O})}_{n;m}\big(\alpha_n;\boldsymbol{\beta}_m\big) \equiv$ generalised function

The form factor axioms

- **Axiom 1** : rapidity exchange

$$\mathcal{F}_n^{(\mathcal{O})}(\beta_1,\ldots,\beta_a,\beta_{a+1},\ldots,\beta_n) \;=\; S(\beta_a-\beta_{a+1})\cdot\mathcal{F}_n^{(\mathcal{O})}(\beta_1,\ldots,\beta_{a+1},\beta_a,\ldots,\beta_n)$$

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- **Axiom 2** : Monodromy

$$\mathcal{F}_n^{(O)}(\beta_1+2i\pi,\beta_2,\ldots,,\beta_n) = \mathcal{F}_n^{(O)}(\beta_2,\ldots,\beta_n,\beta_1)$$

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- **Axiom 3** : Kinematic pole

$$-i\mathrm{Res}\Big(\mathcal{F}_{n+2}^{(\mathcal{O})}(\alpha+i\pi,\beta,\boldsymbol{\beta}_n)\cdot d\alpha\,,\ \alpha=\beta\Big) = \Big\{1-\prod_{a=1}^{n}S(\beta-\beta_a)\Big\}\cdot\mathcal{F}_n^{(\mathcal{O})}(\boldsymbol{\beta}_n)$$

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- **Axiom 4** : Boost

$$\mathcal{F}_n^{(O)}(\boldsymbol{\beta}_n+\Lambda\boldsymbol{e}_n) \;=\; e^{s_O\Lambda}\cdot\mathcal{F}_n^{(O)}(\boldsymbol{\beta}_n)$$

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All "mild" growth solutions $\hookrightarrow$ operator content

⊛ Long history (**’91-’14**) of devising solutions
Zamolodchikov, Fring, Mussardo, Simonetti, Koubek, Lukyanov, Braznikov, Babujian, Karowski, Zapetal ,Feigin, Lashkievich, Pugai, . . .

The p-function representation

$$\mathcal{F}_N^{(\mathcal{O})}(\boldsymbol{\beta}_N) = \prod_{a<b}^{N} \Big\{ \mathrm{F}(\beta_a - \beta_b) \Big\} \cdot \mathcal{K}_N[p_N^{(\mathcal{O})}](\boldsymbol{\beta}_N)$$

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- Minimal form factor

$$\mathrm{F}(\beta) = \exp\left\{-4\int_0^{+\infty} \mathrm{d}x \, \frac{\sinh(x\mathfrak{b}) \cdot \sinh(x\hat{\mathfrak{b}}) \cdot \sinh(\frac{1}{2}x)}{x\sinh^2(x)} \cos\left(\frac{x}{\pi}(\mathrm{i}\pi - \beta)\right)\right\} \quad \text{for} \quad 0 < \Im(\beta) < 2\pi \,.$$

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- Simple system of eqns $\rightsquigarrow$ $p^{(\mathcal{O})}(\boldsymbol{\beta}_N, \boldsymbol{\ell}_N)$

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$$\mathcal{K}_N[p](\boldsymbol{\beta}_N) = \sum_{\boldsymbol{\ell}_N \in \{0,1\}^N} \prod_{a<b}^{N}\left\{1 - \mathrm{i}\frac{(\ell_a - \ell_b) \cdot \sin[2\pi\mathfrak{b}]}{\sinh(\beta_a - \beta_b)}\right\} \cdot \prod_{a=1}^{N}\Big\{(-1)^{\ell_a}\Big\} \cdot p(\boldsymbol{\beta}_N, \boldsymbol{\ell}_N)$$

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- Example ('**98** Braznikov, Lukyanov , '**02** Babujian, Karowski) :

$$p_N^{(\mathrm{e}^{\gamma\boldsymbol{\Phi}})}(\boldsymbol{\beta}_N\mid\boldsymbol{\ell}_N) = \left\{-\mathrm{i}\frac{\mathrm{e}^{\frac{1}{2\pi}\int_0^{2\pi\mathfrak{b}}\frac{t\mathrm{d}t}{\sin(t)}}}{\sqrt{\sin[2\pi\mathfrak{b}]}}\right\}^N\cdot\prod_{a=1}^{N}\left\{\mathrm{e}^{\frac{2\mathrm{i}\pi\mathfrak{b}}{g}\gamma(-1)^{\ell_a}}\right\}$$

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Operator content $\leftrightsquigarrow$ solutions $p_N^{(\mathcal{O})}$

The kernel axioms

- **Axiom 5** : recursive reducibility

$$\mathcal{M}^{(O)}_{n;m}\big(\alpha_n;\beta_m\big) = \mathcal{M}^{(O)}_{n-1;m+1}\big((\alpha_2,\dots,\alpha_n);(\alpha_1+i\pi,\beta_m)\big)$$
$$+\sum_{a=1}^{m}2\pi\delta_{\alpha_1;\beta_a}\prod_{k=1}^{a-1}S(\beta_k-\alpha_1)\cdot\mathcal{M}^{(O)}_{n-1;m-1}\big((\alpha_2,\dots,\alpha_n);(\beta_1,\dots,\widehat{\beta}_a,\dots,\beta_m)\big)$$

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Initialisation :

$$\mathcal{M}^{(O)}_{0;m}(\boldsymbol{\emptyset};\boldsymbol{\beta}_m) = \mathcal{F}^{(O)}_{m;+}(\boldsymbol{\beta}_m)$$

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- Well-defined distributional induction

$\hookrightarrow$ Combinatorial closed form solution

'89 Kirillov, Smirnov , **'24** K., Simon

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Explicit realisation of the fields

III) The correlation functions

Truncated & smeared operators

- Smearing of fields

$$\mathcal{O}[g] = \int_{\mathbb{R}^{1,1}} \mathrm{d}\boldsymbol{x}\, g(\boldsymbol{x})\mathcal{O}(\boldsymbol{x}) \,, \qquad g \in C_c^\infty(\mathbb{R}^{1,1})$$

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- Projection operators $\mathrm{P}_r : \mathfrak{h} \to L^2(\mathbb{R}^r_>)$

$$\mathrm{P}_r \cdot \boldsymbol{f} = (f_0, f_1, \ldots,) = f_r \in L^2(\mathbb{R}^r_>)$$

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- Truncated smeared operators $\mathcal{O}^{(r)}[g] = \mathrm{P}_r \mathcal{O}[g]$

- Truncated regularised correlation functions

$$\left(\boldsymbol{\Omega}, \mathcal{O}_1^{(0)}[g_1]\mathcal{O}_2^{(r_1)}[g_2] \cdots \mathcal{O}_k^{(r_{k-1})}[g_k]\boldsymbol{\Omega}\right), \qquad \boldsymbol{r} = (r_1, \ldots, r_k) \in \mathbb{N}^{k-1}$$

The truncated two-point functions

- Mutually space-like supported $f, g : (\boldsymbol{x}-\boldsymbol{y})^2 < 0$ if $\boldsymbol{x} \in \mathrm{supp}[f]$ and $\boldsymbol{y} \in \mathrm{supp}[g]$

$$\left(\boldsymbol{\Omega}, \mathcal{O}_1^{(0)}[f]\, \mathcal{O}_2^{(N)}[g]\boldsymbol{\Omega}\right) = \int_{(\mathbb{R}^{1,1})^2} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\, \underbrace{\mathcal{W}^{(N)}_{\mathcal{O}_1,\mathcal{O}_2}(\boldsymbol{x},\boldsymbol{y})}_{\mathcal{W}^{(N)}_{\mathcal{O}_1,\mathcal{O}_2}(\boldsymbol{x}-\boldsymbol{y},\boldsymbol{0})} f(\boldsymbol{x})g(\boldsymbol{y})$$

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Formally,

$$\mathcal{W}^{(N)}_{\mathcal{O}_1,\mathcal{O}_2}(\boldsymbol{x},\boldsymbol{0}) = \Big(\Omega, \mathcal{O}_1(\boldsymbol{x}) \mathrm{P}_N \mathcal{O}_2(\boldsymbol{0})\Omega\Big)$$

- Concatenation of integral kernels, $\boldsymbol{x} = (t, x)$

$$\mathcal{W}^{(N)}_{\mathcal{O}_1,\mathcal{O}_2}(\boldsymbol{x},\boldsymbol{0}) = \int_{\beta_1>\cdots>\beta_N} \frac{\mathrm{d}^N\beta}{(2\pi)^N} \underbrace{\mathcal{M}^{(\mathcal{O}_1)}_{0;N}(\emptyset;\boldsymbol{\beta}_N)}_{\mathcal{F}^{(\mathcal{O}_1)}_{N;+}(\boldsymbol{\beta}_N)} \cdot \underbrace{\mathcal{M}^{(\mathcal{O}_2)}_{N;0}(\boldsymbol{\beta}_N;\emptyset)}_{\mathcal{F}^{(\mathcal{O}_2)}_{N;-}(\overleftarrow{\boldsymbol{\beta}}_N + \mathrm{i}\pi\overline{\boldsymbol{e}}_N)} \cdot \prod_{a=1}^{N} \Big\{ \mathrm{e}^{\mathrm{i}m[x\sinh(\beta_a) - t\cosh(\beta_a)]} \Big\}$$

- Symmetrise & deform $\mathbb{R}^N \hookrightarrow \Big(\mathbb{R} + \mathrm{i}\frac{\pi}{2}\mathrm{sgn}(x)\Big)^N$ & boost invariance

$$\mathcal{W}^{(N)}_{\mathcal{O}_1,\mathcal{O}_2}(\boldsymbol{x},\boldsymbol{0}) = \int_{\mathbb{R}^N} \frac{\mathrm{d}^N\beta}{(2\pi)^N N!} \mathcal{F}^{(\mathcal{O}_1)}_N(\boldsymbol{\beta}_N) \Big(\mathcal{F}^{(\mathcal{O}_2^\dagger)}_N(\boldsymbol{\beta}_N)\Big)^* \cdot \prod_{a=1}^{N} \Big\{ \mathrm{e}^{-m\mathfrak{r}\cosh(\beta_a)} \Big\}, \quad \mathfrak{r} = \sqrt{x^2 - t^2}$$

The *per se* two-point functions

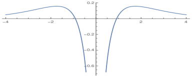
- Full correlation functions $\mathrm{id} = \sum_{N\geq 0} \mathrm{P}_N$

$$\mathcal{W}_{\mathcal{O},\mathcal{O}^\dagger}(\boldsymbol{x},\boldsymbol{0}) = \sum_{N\geq 0} \mathcal{W}^{(N)}_{\mathcal{O},\mathcal{O}^\dagger}(\boldsymbol{x},\boldsymbol{0})$$

- Explicit summand

$$\mathcal{W}^{(N)}_{\mathcal{O},\mathcal{O}^\dagger}(\boldsymbol{x},\boldsymbol{0}) = \int_{\mathbb{R}^N} \frac{\mathrm{d}^N\beta}{N!(2\pi)^N} \prod_{a=1}^{N}\Big\{\mathrm{e}^{-\mathfrak{r}\cosh(\beta_a)}\Big\} \cdot \prod_{a\neq b}^{N}\Big\{\mathrm{e}^{\frac{1}{2}\mathfrak{w}(\beta_a-\beta_b)}\Big\} \cdot \big|\mathcal{K}_N[p_N^{(\mathcal{O})}](\boldsymbol{\beta}_N)\big|^2 \quad , \quad \boldsymbol{\beta}_N = (\beta_1,\dots,\beta_N)$$

- One-body confining potential $\mathfrak{r} > 0$
- Two-body interaction potential ($0 < \mathfrak{b} < \frac{1}{2}$ and $\hat{\mathfrak{b}} = \frac{1}{2} - \mathfrak{b}$)



$$\mathfrak{w}(\lambda) = -4\int_{\mathbb{R}} \mathrm{d}x \, \frac{\sinh(x\mathfrak{b})\cdot\sinh(x\hat{\mathfrak{b}})\cdot\sinh(\frac{1}{2}x)\cdot\cosh(x)}{x\cdot\sinh^2(x)} \mathrm{e}^{\mathrm{i}\frac{\lambda x}{\pi}}$$

- Multi-body interactions

$$\mathcal{K}_N[p_N](\boldsymbol{\beta}_N) = \sum_{\boldsymbol{\ell}_N\in\{0,1\}^N} \prod_{a<b}^{N}\Big\{1 - \mathrm{i}\frac{(\ell_a-\ell_b)\cdot\sin[2\pi\mathfrak{b}]}{\sinh(\beta_a-\beta_b)}\Big\} \cdot \prod_{a=1}^{N}\Big\{(-1)^{\ell_a}\Big\} \cdot p_N(\boldsymbol{\beta}_N,\boldsymbol{\ell}_N)$$

The convergence estimates

Theorem '20, K.

Assume that

$$\big|p_N^{(0)}(\boldsymbol{\beta}_N, \boldsymbol{\ell}_N)\big| \leq C_1^N \cdot \prod_{a=1}^{N} \mathrm{e}^{C_2|\beta_a|^k} \qquad \textit{for some} \qquad C_1, C_2 > 0$$

Then, given any space-like $\boldsymbol{x}$, one has the upper bound

$$\big|\mathcal{W}^{(N)}_{\mathcal{O},\mathcal{O}^\dagger}(\boldsymbol{x}, \boldsymbol{0})\big| \leq \exp\bigg[-\frac{3\pi^4 \mathfrak{b}\,\hat{\mathfrak{b}} \cdot N^2}{4 \cdot (\ln N)^3} \cdot \Big\{1 + \mathrm{O}\Big(\frac{1}{\ln N}\Big)\Big\}\bigg]$$

- The $\sqrt{-\boldsymbol{x}^2}$ dependence is in the control on the remainder.
- Decay is slightly sub-Gaussian : $\frac{1}{N!}$ term is *irrelevant*.
- Fast convergence of form factor expansions

  $\rightsquigarrow$ confirmation of numerical observations
- Proof utilises concentration of measure estimates of N-fold integrals stemming from RMT

Multi-point functions in purely space-like regime

- Partial study of 3 & 4 pt fcts

’00 Balog, Niedermaier, Niedermayer, Patrascioiu, Seiler, Weisz, **’06** Caselle, Delfino, Grinza, Jahn, Magnoli , **’17** Babujian, Karowski, Tsvelik

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Theorem '25, K., Simon

Let $g_1, \dots, g_k \in C_c^{\infty}(\mathbb{R}^{1,1})$ satisfy for $a < b$

$(\boldsymbol{x}_b - \boldsymbol{x}_a)^2 < 0$ & $\boldsymbol{x}_{a;1} > \boldsymbol{x}_{b;1}$ for any $\boldsymbol{x}_c \in \mathrm{supp}[g_c]$, $c \in [\![1 ; k]\!]$.

Then, for $\boldsymbol{r} = (r_1, \dots, r_{k-1}) \in \mathbb{N}^{k-1}$, it holds

$$\big(\Omega, \mathcal{O}_1^{(0)}[g_1]\mathcal{O}_2^{(r_1)}[g_2] \cdots \mathcal{O}_k^{(r_{k-1})}[g_k]\Omega\big) = \int_{(\mathbb{R}^{1,1})^k} \prod_{a=1}^{k} \mathrm{d}\boldsymbol{x}_a \cdot \prod_{a=1}^{k} g_a(\boldsymbol{x}_a) \cdot \mathcal{W}_{\boldsymbol{r}}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k)$$

- Explicit integrand

$$\mathcal{W}_{\boldsymbol{r}}(\boldsymbol{x}_1, \dots, \boldsymbol{x}_k) = \big(\Omega, \mathcal{O}_1^{(0)}(\boldsymbol{x}_1)\mathcal{O}_2^{(r_1)}(\boldsymbol{x}_2) \cdots \mathcal{O}_k^{(r_{k-1})}(\boldsymbol{x}_k)\Omega\big)$$

Preliminary notations

- Concatenated vector

$$\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)\,, \quad \boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_m) \qquad \rightsquigarrow \qquad \boldsymbol{\gamma} \cup \boldsymbol{\vartheta} = (\gamma_1, \ldots, \gamma_n, \vartheta_1, \ldots, \vartheta_m)$$

- Reflected vector

$$\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n) \qquad \rightsquigarrow \qquad \overleftarrow{\boldsymbol{\gamma}} = (\gamma_n, \ldots, \gamma_1)$$

- Vector permutation

$$\mathcal{F}^{(\ell)}(\boldsymbol{\gamma} \cup \boldsymbol{\vartheta}) = \mathrm{S}(\boldsymbol{\gamma} \cup \boldsymbol{\vartheta} \,|\, \boldsymbol{\vartheta} \cup \boldsymbol{\gamma})\, \mathcal{F}^{(\ell)}(\boldsymbol{\vartheta} \cup \boldsymbol{\gamma})$$

- Uniform vector $\boldsymbol{e} = (1, \ldots, 1)$
- Macroscopic momentum

$$\overline{\boldsymbol{p}}(\boldsymbol{\gamma}) = \sum_{a=1}^{n} \boldsymbol{p}(\gamma_a)\,, \qquad \boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_n)$$

The explicit integrand

Explicit density

$$\mathcal{W}_{\boldsymbol{r}}\big(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\big) = \sum_{\boldsymbol{n}\in\mathcal{N}_{\boldsymbol{r}}} \frac{1}{\boldsymbol{n}!} \prod_{b>a}^{k} \Big\{ \int_{\mathscr{C}_{ba}} \mathrm{d}^{n_{ba}}\gamma^{(ba)} \Big\} \cdot \prod_{b>a}^{k} \big\{ \mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\gamma^{(ba)})*(\boldsymbol{x}_b-\boldsymbol{x}_a)} \big\}$$

$$\times \mathcal{S}(\gamma) \cdot \prod_{p=1}^{k} \mathcal{F}^{(O_p)}\Big(\overleftarrow{\boldsymbol{\gamma}^{(pp-1)}} \cup \cdots \cup \overleftarrow{\boldsymbol{\gamma}^{(p1)}} + \mathrm{i}\pi\overline{\boldsymbol{e}}, \boldsymbol{\gamma}^{(kp)} \cup \cdots \cup \boldsymbol{\gamma}^{(p+1p)} \Big)$$

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- Summation domain

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- S-matrix factor $\quad \mathcal{S}(\gamma) = \prod_{\substack{v>p \\ p\geq 3}}^{k-1} \prod_{u>s}^{p-1} S\big(\gamma^{(vu)} \cup \gamma^{(ps)} \mid \gamma^{(ps)} \cup \gamma^{(vu)}\big)$

- Correlation function $\rightsquigarrow$ $\sum_{\boldsymbol{r}\in\mathbb{N}^{k-1}}$ $\hookrightarrow$ totally space-like domains

$$\left(\boldsymbol{\Omega},\mathcal{O}_1(\boldsymbol{x}_1)\mathcal{O}_2(\boldsymbol{x}_2)\cdots\mathcal{O}_k(\boldsymbol{x}_k)\boldsymbol{\Omega}\right) = \sum_{\boldsymbol{n}\in\mathbb{N}^{\frac{1}{2}k(k-1)}} \frac{1}{\boldsymbol{n}!}\prod_{b>a}^{k}\left\{\int_{\mathscr{C}_{ba}} \mathrm{d}^{n_{ba}}\gamma^{(ba)}\right\}\cdot\prod_{b>a}^{k}\left\{\mathrm{e}^{\mathrm{i}\overline{\boldsymbol{p}}(\gamma^{(ba)})*\boldsymbol{x}_{ba}}\right\}$$

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⊛ Convergence ?

- **'24** Guionnet, K., Little Concentration of measure for complex valued β-ensemble oscillatory integrals

  $\hookrightarrow$ first step to attack the problem

- Most Wightmann axioms are "built-in" by construction
  - ⇝ local commutativity is the only difficult to check
- **'89**, Kirillov, Smirnov Weak local commutativity for space-like $\boldsymbol{x}-\boldsymbol{y} : (\boldsymbol{x}-\boldsymbol{y})^2 < 0$

  $\big(\boldsymbol{f}, [\mathcal{O}_1(\boldsymbol{x}), \mathcal{O}_2(\boldsymbol{y})]\boldsymbol{g}\big) = 0$ with $\boldsymbol{f}, \boldsymbol{g} \in \mathfrak{h}$ smooth compactly supported.

  Based on hypothesis of operator product convergent
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Theorem '25, K., Simon

Assume that the series expensions are convergent. Then, $\big(\boldsymbol{\Omega}, \boldsymbol{\Phi}[g_1] \cdots \boldsymbol{\Phi}[g_k]\boldsymbol{\Omega}\big)$ satisfy the Wightman axioms, in particular local commutativity.

If $(\boldsymbol{x}_s - \boldsymbol{x}_{s+1})^2 < 0$ for any $(\boldsymbol{x}_s, \boldsymbol{x}_{s+1}) \in \mathrm{supp}[g_s] \times \mathrm{supp}[g_{s+1}]$

$$\big(\boldsymbol{\Omega}, \boldsymbol{\Phi}[g_1] \cdots \boldsymbol{\Phi}[g_k]\boldsymbol{\Omega}\big) = \big(\boldsymbol{\Omega}, \boldsymbol{\Phi}[g_1] \cdots \boldsymbol{\Phi}[g_{s+1}] \cdot \boldsymbol{\Phi}[g_s] \cdots \boldsymbol{\Phi}[g_k]\boldsymbol{\Omega}\big)$$

IV) Beyond Sinh & Open problems

More general models

- ✔ Throng of solutions to Yang-Baxter equation $\Longrightarrow$ S-matrices of IQFTs

  $\hookrightarrow$ Sine-Gordon, Toda field theories, ...

- ✔ Particle content, form factors (off-shell Bethe Ansatz)

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Open problems

- Convergence of multi-point representations.
- Multi-point functions & Wightmann in more complex models.

  $\rightsquigarrow$ presence of bound states ?
- Extraction from closed formulae of UV behaviour

  $\rightsquigarrow$ prove the Lukyanov conjecture.
- Identification with other construction of Sinh-Gordon (GMC, constr. QFT).