Critical review of constructive QFT

Vieri Mastropietro

University of Rome-Accademia dei Lincei

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$$a_e^{\mathrm{TH}} = 0.00115965218178 \qquad a_e^{\mathrm{EXP}} = 0.00115965218059(13)$$

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- However the mathematical basis of such achievements are still unclear. Divergences, truncation of series non convergent and even not asymptotic (triviality)
- Parallelism with the controversies (Berkeley 1734, Maclaurin 1742,..) on Newton analysis, fluxions, infinitesimals "the ghosts of departed quantites", diverging series, understood after one century of hard work...

BASIC CONCEPTS OF QFT

Euclidean QFT and u.v. or i.r. problems

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- The UV limit is believed to exists for the strong sector of the SM (QCD) but not for the Electroweak sector (EW) or for QED.
- Effective viewpoint in HEP: $1/a$ should be kept finite and large enough so that its effects produce undetectable corrections at the experimental enegies, that is there is a maximum energy scale at which the theory is valid.

QFT: Bosons

- Correlations or Schwinger functions (regularized), from which observables are derived

$$< \Phi_{x_1} ... \Phi_{x_n} >= \frac{\int P(d\Phi) e^{-V(\Phi)} \Phi_{x_1} ... \Phi_{x_n}}{\int P(d\Phi) e^{-V(\Phi)}}$$

where $x \in \Lambda$ lattice $\Lambda = aZ^d \cap [0, L]^d$, a the lattice step. The partition function $\mathcal{Z}$ is the denominator; effective potential $e^{-V_e(\bar\Phi)} = \int P(d\Phi) e^{-V(\Phi+\bar\Phi)}$:generating function $e^{W(\bar\Phi)} = \int P(d\Phi) e^{-V(\Phi)+(\Phi,\bar\Phi)}$.

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- In the Bosonic case ($\Phi = \phi$), $\phi_x \in \mathbb{R}$, $P(d\phi)$ is a Gaussian measure. Example lattice ϕ^4_d, where $P(d\phi) = \frac{1}{\mathcal{N}} \prod_x d\phi_x e^{-a^d Z^\phi_b \sum_x \phi_x(-\Delta+m_b^2)\phi_x))}$, $V = a^d \sum_x [\frac{\lambda_b}{4!} \phi^4_x + \frac{\alpha}{2} \phi^2_x]$, $\lambda > 0$.

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- Pure bosonic theries; Higgs, pure YM, Ising models in $d \geq 3$, ϕ^4....

QFT: Fermions

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- Example Regularized Fermi model in $d = 4$ (similar in $d = 2, 3$)

$$P(d\psi) = \frac{1}{\mathcal{N}} \prod_x d\bar\psi_x d\psi_x e^{-a^4 Z_b^\psi \sum_x \bar\psi_x (D\psi_x))} \qquad V = \lambda_b \sum_x j_{\mu,x} j_{\mu,x}$$

$(D\psi)_x = \sum_y D_{x,y}\psi_y$ is the lattice Dirac derivative, in Fourier space $(\hat g(k))^{-1} = \frac{1}{a}\sum_\mu (\gamma_\mu \sin p_\mu a + 2\sin^2 a p_\mu/2 + m_b)$,$\mu = 0, 1, 2, 3$ where γ_μ are the gamma matrices and $j_{\mu,x} = \bar\psi_{\mu,x}\gamma_\mu\psi_{\mu,x}$; Wilson term $2\sin^2 a p_\mu/2$ to cure the fermion doubling due to the extra poles in addition to 0. Formal "$\int \mathcal{D}\psi e^{-\int dx(\bar\psi(\partial\!\!\!/+m)\psi - \lambda j_\mu j_\mu}$".

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- $\mathcal{E}(\prod_i \psi_{x_i}\bar\psi_{y_i}) = \sum_\pi \varepsilon_\pi \prod_{(i,j)\in\pi} g(x_i, y_j)$, where ε_π is the sign of the permutation. Again by this a perturbative expansion is obtained.

QFT: Boson-Fermion

- QFT for boson-fermion systems: QED, Electroweak theory, QCD. Ex. regularized massive QED$_4$(or in $d = 2, 3$) (mass could be generated by Higgs), $\Phi = \psi, \phi$

$$P(d\Phi) = P(d\psi)P(dA) \quad P(dA) = \frac{1}{\mathcal{N}} \prod_x dA_x e^{-a^4 Z_b^A \frac{1}{L^4} \sum_k \hat{A}_{\mu,k} (\hat{g}_{\mu\nu})^{-1} \hat{A}_{\mu,k})}$$

with $\hat{g}^A_{\mu,\nu}(k) = \frac{1}{Z^A} \frac{1}{|\sigma|^2 + M^2} (\delta_{\mu,\nu} + \frac{\bar{\sigma}_\mu \sigma_\nu}{M^2})$, $\sigma_\mu = (e^{ip_\mu a} - 1)/a$.

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- Formally " $\int \mathcal{D}\psi\mathcal{D}A e^{-\int d^dx\left(\frac{1}{4}F_{\mu,\nu}F_{\mu,\nu}+\frac{1}{2}M^2A_\mu A_\mu+\bar{\psi}(i\partial\!\!\!/+m+ieA_\mu)\psi+\frac{1}{2}\xi(\partial_\mu A_\mu)^2\right)}$ ";
invariance under $\psi\to\psi e^{i\alpha}$ implies $\partial j_\mu=0$ (current conservation); if $m=0$ $\psi\to\psi e^{i\gamma_5\alpha}$ implies $\partial j^5_\mu=0$ (axial current conservation) $j^5_\mu=\bar{\psi}\gamma_\mu\gamma_5\psi$. ξ is the Gauge fixing parameter: physical observables are ξ independent (hence one can add the ξ). If $M=\xi=0$ invariant under $A_{\mu,x}\to A_\mu+\partial_\mu\alpha_x$, $\psi_x\to\psi_x e^{i\alpha_x}$ (Gauge invariance).

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- We can add a ξ term to the lattice model so the the longitudinal part is $\frac{(1-\xi)\bar\sigma_\mu\sigma_\nu}{(\xi|\sigma|^2+M^2}$) or $\frac{(1-1/\xi)\bar\sigma_\mu\sigma_\nu}{(|\sigma|^2}$) in $\hat g^A_{\mu,\nu}(k)$.

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- With other regularizations not true, like momentum there is no WI and ξ independence. WI and ξ independence are crucial for physical significance.
- If $m = 0$ Axial current is not conserved chiral anomalies $\partial_\mu j^5_\mu = \frac{e^2}{16\pi^2}\varepsilon_{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$. In EW sector the interaction is chiral(mass must be generated by Higgs) and it is not known a regularization ensuring WI and ξ independence; it should exist only for certain parameters (anomaly canc. cond.). Yang-Mills extension to $SU(n)$.

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- Criterion of renormalizability by the degree of divergence (power counting) $D_{n_e,n}$ associated to to Φ^{n_e} in V_e of order n; if is independent on n is renormalizable. Non trivial to prove at any order Bogolubov 1957, Hepp 1966. In ϕ^4 $D_{n_e,n} = (d-4)n + d - (d-2)n_e/2$ If $d = 4$ renormalizable ($d \leq 3$ superrin, $d \geq 5$ non-rin). Fermi model $D_{n_e,n} = (d-2)n + d - (d-1)n_e/2$; non ren in $d = 4, 3$, ren in $d = 2$.

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- True with charges $(0, -1, 2/3, -1/3)$ if there is the anomaly non-renormalization (Adler-Bardeen 1966). Explainy why proton and electron exactly opposite charges.

Perturbative series

- At the end if renormallizability is present one can compute and get finite quantities by truncation. For instance the contribution of the Electroweak (EW) theory to g is (e electric charge, g_w weak constant, $\alpha = e^2/\hbar c \sim 1/137$) $a_{EW} = \sum_{n,m=1}^{\infty} A_{n,m} e^n \mathrm{g}_{\mathrm{w}}^m$

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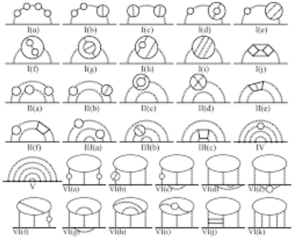
- $A_{2,0} e^2 = \frac{\alpha}{\pi}$ $\quad A_{4,0} e^4 = -(\frac{\alpha}{\pi})^2 0.328...,$ $A_{0,2} \mathrm{g}_{\mathrm{w}}^2 = a_{\mathrm{Z},1} + a_{\mathrm{W},1},$ $\quad a_{\mathrm{Z},1} = \frac{m^2}{M_Z^2} \frac{g_W^2}{4\pi^2} B$ with B a function of M_W/M_Z; m electron mass; $A_{1,0}$ Schwinger (1948), $A_{2,0}$ Sommerfeld (1958), $A_{0,2}$ Jackiw Weinberg (1972),...Aoyoama (2012) 10-th order. Feynman graphs. (Why is the truncation justified?)



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- The terms with $D_{n_\phi,n_\psi}>0$ are increased relevant; if $D_{n_\phi,n_\psi}<0$ are decreased irrelevant; if unchanged by this scaling argument marginal.

- The integration of the field Φ^h is done separating the irrelevant terms from the others, and writing the marginal or relevant terms as local ones; quadratic inserted in gaussian part
$$\int P(d\Phi)e^{V(\Phi)} = \int P(d\Phi^{\leq h})e^{-\mathcal{L}V(\Phi^{\leq h})-\mathcal{R}V(\Phi^{\leq h})}$$
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- In HEP one choosen parameters at scale N to have a fixed value at scale 0 (chosen by experiments). In cond mat one starts from $N = 0$. The irrelevant terms decrease but have anyway a crucial role.

Renormalization Group

- In a non-renormalizable theory like Fermi theory $d = 4$, $D_4 = -2$ the coupling decrease as $\lambda_b \gamma^{2(h-N)}$ so if $\lambda_d = \lambda_0 = O(1)$ then $\lambda_b = O(\lambda_d \gamma^{2N})$, hence maximal cut-off $\gamma^{2N} \sim 1/\lambda_d$. In cond mat $N = 0$ λ_h becomes smaller and smaller $h \leq 0$. In a superrenormalizable (like ϕ^4_3) $D_4 = 1$ the coupling increases as $\lambda_b \gamma^{-(h-N)}$ hence then $\lambda_d = \lambda_0 = O(\lambda_d \gamma^N)$, hence $\lambda_d = O(\lambda_b \gamma^{-N})$.

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- In QCD the beta function has the opposite sign and the bare coupling $\to 0$ as $N \to \infty$. Asymptotic Freedom. There the cut-off could be removed.

A short recap

- Infinities require the introduction of cut-off, to be removed or at least very large. Relation between renormalizability and cut-off; in a renormalizable theory the cut-off could be exponentally high while in a non renormalizable is large as a power of the inverse coupling.

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- QED is renormalizable. Dyson (1949), Tomonaga Schwinger Feynman. One could reach exponentially high cut-off (no asymptotic fredom). Renormalizability due to Ward Identities. Need relations between rcc; WI broken by several RG scheme or momentum regularizations. Impressive $g-2$ prediction.

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- Weak forces; adding electron and boson mass ”by hands” breaks renormalizability and WI (due to chiral forces). Electroweak theory Weinberg (1967), mass generated by Higgs. Renormalizability ’t Hooft (1971); crucial that the longitudinal part cancel (decrease of divergence degree). True only if anomalies cancel Bouchiat, Iliopoulos, Meyer (1972) based on Adler-Bardeen theorem (1967). Not known regularization ensuring WI at cut-off finite.

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- Several prediction in Cond Mat and stat phys, BCS, Hall, exponents, ...

Part II

SOME CLASSICAL RESULTS IN MATHEMATICAL PHYSICS

RG and trees

- The Wilson RG was used by Gallavotti RMP 1985, Gallavotti and Nicolo' CMP 1983 to write correlations in ϕ^4_4 (and related models) as series in the rcc $\sum_n a_n v_h^n$ with concrete bounds $|a_n| \leq C^n n!$ with C independent on h (this correspond to the proof of perturbative renormalizability using RG).

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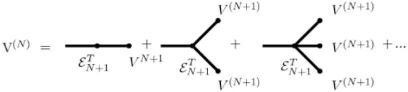
- This was based on the GN-tree expansion. $V^N = \sum_n \frac{1}{n!}\mathcal{E}^T_N(V;n)$



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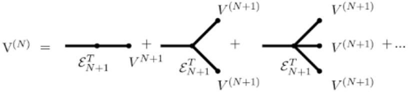

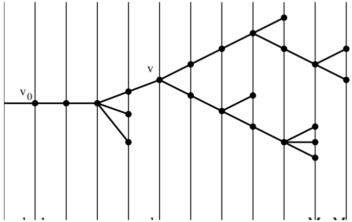

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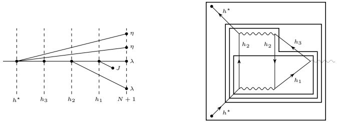

Figure 1: Fourth order diagram with the same structure as [**Karplus_Knoll_1950**], together with corresponding tree.

RG and flow equations

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- The WI are recovered order by order removing cut-off (not expected non perurbatively). Extended to YM in Kopper CMP 1999

Basic non perturbative results: fermions

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- Not used in the first non perurbative construction of a rernormalizable model with i.r. cut-off, the GN_2 model which is asymptotically free, $h \geq 0$, $\lambda_h \sim \frac{\lambda_0}{1-\lambda_0 h}$. Gawedzki, Kupiainen (CMP 1985) and in Feldman, Magnen, Rivasseau, Seneor (CMP 1987): phase cell decomposition and Mayer and cluster expansion.

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- Gawedzki Kupiainen Nucl Phys (1985) defined an epsilon expansion for fermions in the uv. Epsilon expansion for the ir and analyticity of exponents and fixed point for simplectic fermions with non-local kinetic energy by Giuliani Mastropietro Ryckov Scola CMP 2024. Application in cond mat, Jellium or Hubbard finite T in 2d, ext FS.

Non perturbative bosons: ϕ^4 and YM

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- Several recent results in the collaboration of this conference for bosons and fermions

QED or EW

- The cut-off cannot be removed in EW sector of the SM (triviality, in ϕ^4 Froelich 1982 $d > 4$ Aizenman Duminil Copin (2019) $d = 4$. The main mathematical problem is here to construct the theory up to a large exponential cut-off for "reasonable" parameters (eg $m/M_Z \sim 10^{-6}$, $\alpha \sim 1/137$ very small). Eg explain $g - 2$. Largely unsolved even for QED : only Fermi theory exists but there the cut-off is a power law.

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- decomposition of bosons and fermions. Control of irrelevant quantities as $g - 2$ and non renormalization of anomalies. Let us see some results in this direction.

SOME RECENT RESULT VIA RENORMALIZATION GROUP

Massive QED2

- Paradigmatic model; Massive QED2 . The formal continuum model is $S(A,\psi)=\int d^dx\,\left(\frac{1}{4}F_{\mu,\nu}F_{\mu,\nu}+\frac{1}{2}M^2A_\mu A_\mu+\psi(i\partial+i\mathsf{e}A_\mu)\psi+\frac{1}{2}\xi({}_\mu A_\mu)^2\right)$

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- Perturbatively with dimensional regularization (Georgi-Rawls 1971) $p_\mu\langle \hat{\jmath}_{\mu,p};\hat{\jmath}_{\nu,-p}\rangle = 0$ and $p_\mu\langle \hat{\jmath}^5_{\mu,p};\hat{\jmath}_{\nu,-p}\rangle = \frac{i}{\pi}\varepsilon_{\mu,\nu}p_\mu$, ξ-independence, finite bare parameters; decrease of divergence degree; from renormalizable to superrenormalizable; Adler-Bardeen non-renormalization of anomaly. Toy model for QED or EW.

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- Formal exact solutions (Sommerfield 1962, Hagen 1967). Vanishing bare wave function renomalization (anomaly) $Z=0^\eta$ with $\eta>0$ and anomaly $\zeta\frac{1}{\pi}\varepsilon_{\mu,\nu}p_\mu$; depends on the regularization ($\zeta=1/2$ with momentum). Different from perturbative results.

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- Lattice regularization with Wilson term (not solvable)

$$e^{W(J,J^5,\phi)}=\int P(dA)\int P(d\psi)e^{V_c(\psi)+V(A+J,\psi)+B(\psi,J^5,\phi)}$$

Massive QED2

- V is obtained by $P(d\psi)$ replacing $\bar\psi_{x\pm e_\mu}(e^{\pm i e_b \sqrt{Z^A_b} a A_{\mu,x}}-1)\psi_x$, $V_c=\nu a^2\sum_x \bar\psi\psi$, B source, $j^5_\mu=Z_5\bar\psi\gamma_\mu\gamma_5\psi$; ν mass counterterm WI and ξ independence.The constant Z_5 is fixed requiring the charge of j^5 is the same as j (amputated $< j\psi\psi >$).

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- Theorem*If* $e^2\le\varepsilon_0$ *independent on* L,a, *for suitable* ν, Z_5 *(massless case) the correlations are given by convergent expansions, the limit* $L\to\infty$ *exists,* $<\bar\psi_k\psi_k>$ *diverges as* $1/|k|^{1+\eta}$, $\eta=O(e^2)$ *and*

$$p_\mu\langle\hat j^5_{\mu,p};\hat j_{\nu,-p}\rangle=\frac{1}{\pi}\varepsilon_{\nu,\mu}p_\mu+O(|p|^2)$$

.

The result holds for arbitrarily small lattice steps and L,a (uniformly in cut-off ir or uv, limits can be taken); non renormalization of anomaly. There is no wave function renormalization and reduction of divergence degree. Relation between the rcc due to WI at each step. Same feature as in EW or QED. Mastropietro, Phys Rev D 2022, based on Fabbri, Mas., Renzi arxiv 2602.03705 and Benfatto Mas. CMP 2004.

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- Two regimes: the uv $\gamma^h \geq M$ and the i.r. $\gamma^h \leq M$. In the uv cancellations by partial decomposition of determinants extracting propagators without generating $n!$ in BFF formula. Essential ξ independence (WI,reduction of divergence degree). The lattice produce an effective interaction non quartic; superrenormalizable, like a lattice Y2.

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- In the i.r. regime renormalized GN tree expansion in terms of rcc (renormalizable) that is λ_h (ψ^4), Z_h $(\psi\partial\psi)$, Z_h^J $(A\psi\psi)$. Singular wave function $Z_h \sim_{h\to\infty} \gamma^{-\eta h} \to \infty$. $\lambda_0 \sim e^2$.

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- Anomalous integration; rescale at each integration step the field $\psi/\sqrt{Z_h}$. It works only if the rcc are not independent (similar to QED_4 $Z_h^1 = Z_h^2$).

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- In addition the anomaly get contributions also from the irrelevant terms; they need to cancel at every order to get the AB property.

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- It is a regularized non local Thirring , model. It is the "fixed point" for a large class of models $x \in [0, L)$ no lattice

$$\int P(d\psi^{(\leq N)}) e^{\tilde{\lambda}\tilde{Z}^2\tilde{\lambda}\int dxdy v(x-y)\rho_{+,x}\rho_{-,y} + \sum_\omega \tilde{Z}^+ \int dx J^+_{\omega,x}\rho_{\omega,x} + \tilde{Z}^- \int dx J^-_x \omega\rho_{\omega,x}}$$

with $\psi = \psi_1, \psi_{-1}$, $\hat{g}_\omega(k) = \frac{1}{Z}\frac{\chi_N(k)}{-uk_0+\omega k}$, $\omega = \pm$ $\chi_N(k)$ is a smooth non vanishing for $\gamma^h \leq |k| \leq \gamma^N$, $v(x-y)$ a short range symmetric potential with range κ (regularized δ) and $\rho_{\omega,\mathbf{x}} = \psi^+_{\omega,x}\psi^-_{\omega,x}$. Constructed in Benfatto Mastropietro CMP 2004 (even $\kappa \to 0$, diverging $Z \sim \kappa^{-\eta}$)

- Two regime infrared and ultraviolet. The infrared very similiar to the massive QED_2; same rcc. If the flow of the rcc can be controlled in this model is also the QED_2.

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- The momentum cut-off produces a violation of WI One obtains, $D_\omega(k) = -ik_0 + \omega k_1$

$$D_\omega < \hat\rho_{p,\omega} \hat\psi^+_{k,\omega'} \hat\psi^-_{k+p,\omega'} > + \Delta_N(k,p) = \delta_{\omega,\omega'} \frac{1}{Z} [< \hat\psi^+_{k,\omega'} \hat\psi^-_{k,\omega'} > - < \hat\psi^+_{k+p,\omega'} \hat\psi^-_{k+p,\omega'} >$$

where $\Delta_N =< \delta\hat\rho_{p,\omega} \hat\psi^+_{k,\omega'} \hat\psi^-_{k+p,\omega'} >$ with

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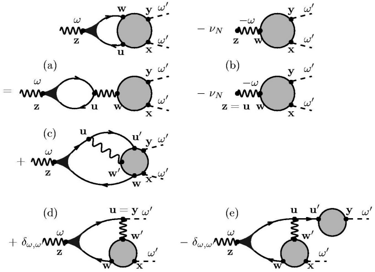
- We can analyze the Δ_N by RG. It is not vanishing even if $-h, N \to \infty$ and one gets

$$\lim_{-h,N\to\infty} \Delta_N(k,p) = \tau \hat v(p) D_{-\omega}(p) < \hat\rho_{p,-\omega} \hat\psi^+_{k,\omega'} \hat\psi^-_{k+p,\omega'} >$$

with $\tau = \frac{\tilde\lambda}{4\pi v}$. Both the current and the axial current are not conserved but have a simple form.



The δj term has at least a field at scale N; in all the terms exept the first there is a gain. The bubble with the δj term gives $1/4\pi$.

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- Combined with Schwinger.-Dyson equation gives the needed informations for the rcc at scale h $\lambda_h \sim \lambda Z_h^2$; this implies vanishing of the beta functions also for the lattice model.
- In addition one has in the reference model exact relations in the limit $-h, N \to \infty$ exact WI for current and chiral current, if $\tilde\Gamma^+_\mu =< j_\mu \bar\psi\psi >, \tilde\Gamma^-_\mu =< j^5_\mu \bar\psi\psi >, \tilde S =< \bar\psi\psi >, \tilde\Gamma^5_{\mu,\nu} =< j^5_\mu j_\nu >$

$$(1 \mp \tau)p_\mu \tilde\Gamma^\pm_\mu(k, p) = \frac{\tilde Z^\pm}{\tilde Z}(\tilde S(k) - \tilde S(k + p)) \quad p_\nu \tilde\Gamma^5_{\mu,\nu} = \frac{\tilde Z^+ \tilde Z^-}{4\pi Z^2}\frac{\varepsilon_{\nu\mu} p_\nu}{(1 - \tau)}$$

$p_\mu \tilde\Gamma^5_{\mu,\nu} = \frac{\tilde Z^+ \tilde Z^-}{4\pi Z^2}\frac{\varepsilon_{\mu\nu} p_\mu}{(1+\tau)}$. Note that both the WI have anomalies (due to momentum cut off).

- The 2 models differs by irrelevant terms; one can choose $\tilde{Z}, \tilde{\lambda}, \tilde{Z}^{\pm}$ so that the rcc becomes asymptotically equal, if $\hat{\Gamma}^5_{\mu,\nu}(p)$ is the current-current in massive QED$_2$

$$\hat{\Gamma}^5_{\mu,\nu}(p) = Z_5 \tilde{\Gamma}^5_{\mu,\nu}(p) + R_{\mu,\nu}(p)$$

where now $R_{\mu,\nu}(p)$ is continuos while $\hat{\Gamma}^5_{\mu,\nu}(p)$ is not. Instead Γ_μ, S differ by subleading terms.

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- Finally $p_\nu \hat{\Gamma}^5_{\mu,\nu}(p) = \frac{\tilde{Z}^+ \tilde{Z}^-}{2\pi \tilde{Z}^2} \frac{\varepsilon_{\nu\mu} p_\nu}{(1-\tau)} + p_\nu R_{\mu,\nu}(p) = 0$ we get $R_{\mu,\nu}(0) = -\frac{\tilde{Z}^+ \tilde{Z}^-}{2\pi Z^2} \frac{\varepsilon_{\mu\nu}}{(1-\tau)} = -(1+\tau)\varepsilon_{\nu\mu}/Z_5$ (crucial that $R(p)$ is continuous) hence $p_\mu \hat{\Gamma}^5_{\mu,\nu}(p) = Z_5 p_\mu [\tilde{\Gamma}^5_{\mu,\nu}(p) + R_{\mu,\nu}(p)] = [(1-\tau)\varepsilon_{\mu,\nu} - (1+\tau)\varepsilon_{\nu,\mu}] p_\mu / 4\pi = 1/2\pi \varepsilon_{\mu,\nu} p_\nu$ that isall the dependence of the coupling disappears up to subleading terms .

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- Recent result (Bianchessi Mas Porta);extend to several flavors and prove $\partial_\mu j^5_\mu = 0$ under $\sum e_L^2 = \sum e_R^2$ in a lattice ; there are graphs mixing flavors but they cancel.

Lattice fermion-vector model

- We can consider massive lattice QED$_4$; Adler-Bardeen (1966) say order by order $p_\mu \Pi^5_{\mu,\rho,\sigma} = -\frac{1}{12\pi^2}\varepsilon_{\alpha,\beta,\nu,\sigma} p^1_\alpha p^2_\sigma + e^2 0 + e^4 0 + \ldots$ if $m = 0$ absence of radiative corrections (formally without cut-off) ($\Pi^5_{\mu,\rho,\sigma} =< j_\mu j_\rho j_\sigma >$)-perturbative proof based on dimensional regularization; diverging series. Integrating out the boson field again one reduces to a fermionic theory (essentially Fermi theory with cutoff-irrelevant).

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Adler Bardeen non-renormalization at finite lattice.
- The construction holds up to cut-off of the $O(1/e^2)$ (Fermi theory is analytic). Crucial role of irrelevant terms. It is an essential ingradient for the SM with cut-off

Anomalies in 4d QFT

- By multiscale RG analysis one gets the following decomposition in a more and less regular part

$$\hat{\Pi}^5_{\mu,\nu,\sigma}(p_1,p_2) = \frac{Z^5_\mu Z_J Z_J}{Z^3} I_{\mu,\nu,\sigma}(p_1,p_2) + \hat{H}^5_{\mu,\nu,\sigma}(p_1,p_2)$$

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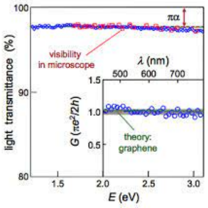
- Expanding H at first order and fixing derivatives by current conservation. $\sum_\nu p_{1,\nu} \hat{\Pi}^5_{\mu,\nu,\sigma}(p_1,p_2) = 0$ we compute $p_\mu \Pi^5_{\mu,\rho\sigma}$ using the decomposition of Π^5 and get the anomaly non renormalization

- A similar argument can be used to prove the exact quantization of Hall conductivity $\sigma = ne^2/h$ or the universlity of transport coefficients in graphene $\sigma = e^2/h(\pi/2)$. Giuliani Mastropietro Porta CMP 2011, CMP 2015. Exact independence from the interaction. Graphene Geim e al 2008



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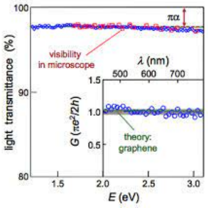

- Close argument implies cancellation of anomalies (Mastropietro AHP 2022) with cut-off $O(1/e^2)$ in a lattice chiral gauge theory with condition $\sum_i e_{i,L}^3 - \sum_i e_{i,R}^3$; in this regime the only contributing is the triangle but with higher cut-off some cancellation is needed (but found in $d = 2$)

Higher cut-offs

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- One produces couplings Z_h $(\bar\psi \partial \psi)$, Z^1_h $(A_\mu j_\mu)$, Z^A_h $(A\partial^2 A)$, M_h (A^2), Z^4_h (A^4), m_h (ψ^2) ; by proceeding as in QED_2 considering a reference model with an infrared cut-off h and deriving WI for each h using that the scales $\leq h$ are superrrenormalziable one get relations $Z^1_h \sim Z_h$, $e_h \sim 1/Z^A_h$ and M_h, Z^4_h bounded in h. No counterterms: difference between Feldman et al in QED_4 or Magnen Rivasseou in YM_4.

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- Done order by order (Giuliani, Fabbri, Falconi, Mastropietro 2025)-hopefully control on large fields. Need to prove non reonormalizazion of anomaly in renormalizable regime (maybe needs marginal irrelevance? or cancellations as in 2d-issue related to graphene).

$g-2$

- Finally let us return to $g-2$; consider the Standard Model as an effective theory valid up to certain energy scales and therefore requiring an energy cutoff, and prove that it can be taken large enough so that it agrees the perturbative computations at lowest order, up to a mall error, in a physical range. Λ large to see no effect of it but not too much to see divergence.

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- Perturbatively $a=(g-2)/2$, contribution EW $a=\sum_{n,m} A_{n,m} e^n \mathrm{g}_{\mathrm{w}}^m$ $\alpha = e^2/(\hbar c) = 1/137, ..$ and $\mathrm{g}_{\mathrm{w}}^2 = 4\pi\alpha/\sin^2\theta_W$ with $\sin^2\theta_W = 0,232...$ Weak contributions. Jackiw Weinberg (1972) $A_{0,2}\mathrm{g}_{\mathrm{w}}^2 = a_{\mathrm{z},1} + a_{\mathrm{w},1}$ and

$$a_{\mathrm{z},1} = \lambda^2 \frac{m^2}{M^2}\frac{1}{4\pi^2}\frac{1-5\kappa^2}{3}\left[1+O\left(\frac{m^2}{M^2}\right)\right] = a_{\mathrm{z},1}\left[1+O\left(\frac{m^2}{M^2}\right)\right]$$

where $\lambda = \mathrm{g}_{\mathrm{w}}(4\sin^2\theta_W - 1)/(4\cos\theta_W), \kappa = (4\sin^2\theta_W - 1)^{-1}$, $M = M_Z$ The contribution of weak forces is much smaller than e.m. by a factor $m^2/M^2 \sim 10^{-6}$.

Neutral contribution to $g-2$

- Contribution of the weak Z boson to $g-2$ (similar to QED but with a massive boson) Regularization with finite cutoff Λ; the contribution to $g-2$ is a_Z^R $(\lambda = g_W)$.

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- Remark: g is used to compute α, as transport coefficients like graphene or Hall conductivity. In one case is a truncated series, in the other is believed exact expression. Why? RG explains this, see below

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- Mastropietro Phys. Rev. D 2024; Mastropietro Bianchessi 2025 arxiv510.02092
  Theorem *For* $\lambda^2 \leq C^{-1}(M^2/\Lambda^2)\log^{-2}(M/m)$ *in the infinite volume limit*

$$a_Z^R = a_{Z,1}(1+R_\lambda) \qquad |R_\lambda| \leq C_0 \frac{m^2}{M^2} + C_1 \frac{M^2}{\Lambda^2} + C_2 \frac{\lambda^2 \Lambda^2}{M^2} \cdot \log^4\left(\frac{M}{m}\right)$$

with $a_{Z,1} = \frac{m^2}{M^2}\frac{\lambda^2}{4\pi^2} B$ *the perturbative Jackiw–Weinberg result and* C_i *constants.*

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- One need M/Λ small to make the second term small; then the third terms is small if $\lambda << (M/\Lambda)\log^{-1}(M/m)$. Upper and lower bound.

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- One would like to improve to $\lambda < \log^{-1}(\Lambda/M) \log^{-2}(M/m))$ i.e. exponentially large cutoffs In this case the WI are not preserved even with the lattice, anomaly should cancel and relations between rcc must be found.

Trees and diagrams

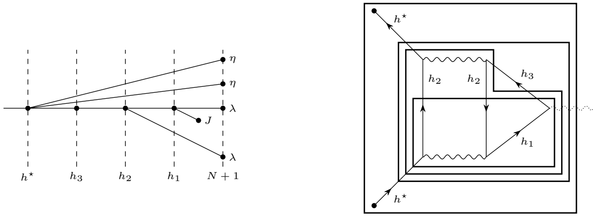

Figure 1: Fourth order diagram with the same structure as [**Karplus_Knoll_1950**], together with corresponding tree.

$$||W|| \leq \underbrace{\left(\frac{C\lambda^2}{M^2}\right)^2}_{\text{Endpoints}} \cdot \underbrace{\gamma^{-h^\star - h^\star}}_{h^\star \text{ cluster}} \cdot \underbrace{\gamma^{3h_3}}_{h_3 \text{ cluster}} \cdot \underbrace{\gamma^{3h_2 - h_2}}_{h_2 \text{ cluster}} \cdot \underbrace{\gamma^{-h_1}}_{h_1 \text{ cluster}} \cdot \underbrace{\gamma^{2(h^\star - h_3)}}$$

$$||W|| \leq \left(\frac{C\lambda^2\Lambda^2}{M^2}\right)^2 \cdot \gamma^{2(h_1-N)+2(h_2-N)-2h^\star} \cdot \gamma^{(h_3-h^\star)\cdot(0-2)+(h_2-h_3)\cdot(-3-0)+(h_1-h_2)\cdot(-3-0)},$$

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- hence we are still on schedule for QFT