Conformal Bootstrap of Probabilistic Models in $d \geq 3$

Dalimil Mazáč
IPhT + IHES

Courant Institute, Apr 27 2026

conformal bootstrap = axiomatic approach to CFT_d

This talk :

- What is the conformal bootstrap in d dimensions ?
- How does it relate to the probabilistic approach ?
- Bonus: analogy with spectral theory of automorphic forms.

Plan:

1. CFT_d s as measures on spaces of distributions

2. reflection positivity & the bootstrap

3. the bootstrap without reflection positivity

- Classically, CFTs arise from statistical lattice models.
- Fundamental example: Ising model on $\mathbb{Z}^d$ $(d \geq 2)$
- Prob. measure on maps $\sigma : \{-N, \dots, N\}^d \to \{-1, 1\}$

$$\mathbb{P}(\sigma) = \frac{1}{\mathcal{Z}} \exp\left[-\beta H(\sigma)\right] , \qquad H(\sigma) = - \sum_{(x,y) \in E} \sigma(x)\sigma(y)$$

- $\sigma = +1$ on $\partial\{-N, \dots, N\}^d$

  $N \to \infty \Rightarrow$ prob. measure on $\{-1, 1\}^{\mathbb{Z}^d}$

$\exists \ \beta_c > 0$ s.t.

- $\beta < \beta_c \Rightarrow \mathbb{E}_\beta(\sigma(x)\sigma(y)) = O(\exp(-\# |x-y|))$ as $|x-y| \to \infty$

AIZENMAN, BARSKY, FÉRNANDEZ

- $\beta = \beta_c$ expect $\mathbb{E}_{\beta_c}(\sigma(x)\sigma(y)) \sim \dfrac{A(d)}{|x-y|^{2\Delta(d)}}$ as $|x-y| \to \infty$

$d = 2 : \quad \Delta = \dfrac{1}{8}$ ONSAGER, ...

$d = 3 : \quad \Delta \approx 0.518148806(24)$ RYCHKOV, POLAND, SIMMONS-DUFFIN, ...

$d \geq 4 : \quad \Delta = \dfrac{d-2}{2}$ FRÖHLICH, SIMON, SPENCER, HARA, SLADE, AIZENMAN, DUMINIL-COPIN, ...

Scaling limit @ $\beta = \beta_c$

- Fix $a > 0$, let $$\phi_a := \frac{1}{\sqrt{A(d)}} a^{d-\Delta(d)} \sum_{x \in \mathbb{Z}^d} \sigma(x) \delta_{ax}.$$

- ϕ_a is a random distribution on $\mathbb{R}^d$.

Conjectures:

1) $\phi := \lim_{a \to 0^+} \phi_a$ exists (as a random distribution)

2) The resulting measure $\mu(d)$ on $\mathcal{S}'(\mathbb{R}^d)$ is universal.

3) $\mu(d)$ is conformal invariant.

- self-similarity under scaling KADANOFF, WILSON
- POLYAKOV '70: "scale invariance + local interactions $\Rightarrow$ conformal invariance"

<u>Definition</u>: Let (M, g), $(\tilde{M}, \tilde{g})$ be Riemannian manifolds

A diffeomorphism $\alpha : M \to \tilde{M}$ is **conformal** if

$\alpha^* \tilde{g} = \Omega_\alpha^2 g$ for some $\Omega_\alpha : M \to \mathbb{R}_+$.

<u>Theorem (Liouville)</u>: For $d \geq 3$, any conformal diff. between domains in $\mathbb{R}^d$

extends to a conformal diff. $S^d \to S^d$.

<u>Theorem</u>: For $d \geq 2$, $\mathrm{Conf}^+(S^d) \cong SO_0(1, d+1)$.

Let $f \in C^\infty(S^d)$, $\alpha \in \mathrm{Conf}^+(S^d)$, $\phi \in \mathcal{D}'(S^d)$

$$(\alpha \cdot \phi)(f) := \phi\left(\Omega_\alpha(\cdot)^{d-\Delta} f(\alpha(\cdot))\right). \qquad (*)$$

Definition: A measure on $\mathcal{D}'(S^d)$ preserved by $(*)$ is Δ-conformal invariant.

Conjecture: The scaling limit of Ising_d @ β_c is conformal invariant.

- $d = 2$ ✓ CAMIA, GARBAN, NEWMAN '13
- $d = 3$ open
- $d \geq 4$: Gaussian free field ✓

conformal invariance of GFF_d :

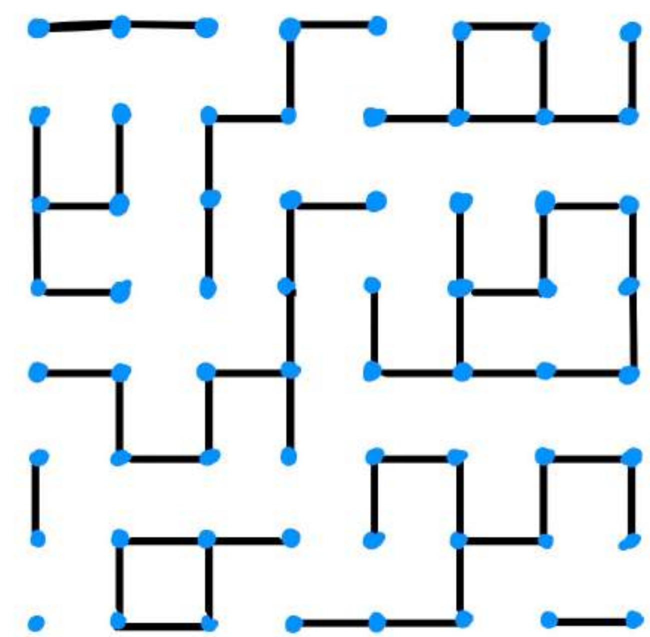
GFF_d : $\forall f \in \mathcal{S}(\mathbb{R}^d)$ $\phi(f)$ is a Gaussian r.v.

with $\mathbb{E}(\phi(f)) = 0$ & $\mathbb{E}(\phi(f_1)\phi(f_2)) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{f_1(x) f_2(y)}{|x-y|^{d-2}} \, dx dy$.

conf. inv. $\Leftrightarrow$ invariance of covariance kernel $\frac{1}{|x-y|^{d-2}}$.

$\left(\Delta = \frac{d-2}{2} \right)$.

Bernoulli percolation on $\mathbb{Z}^d$



- Edges are i.i.d. random variables
- $\mathbb{P}(\text{open}) = p$, $\mathbb{P}(\text{closed}) = 1-p$.
- Let $x \sim y$ denote the event that $x, y \in \mathbb{Z}^d$ are connected by open edges.
- Critical $p = p_c$ s.t. $p \le p_c$: $\mathbb{P}(x \sim \infty) = 0$

  $p > p_c$: $\mathbb{P}(x \sim \infty) > 0$

Conjecture: 1) At $p = p_c$ $\quad \mathbb{P}(x \sim y) \sim \dfrac{B(d)}{|x-y|^{2\Delta'(d)}}$ as $|x-y| \to \infty$

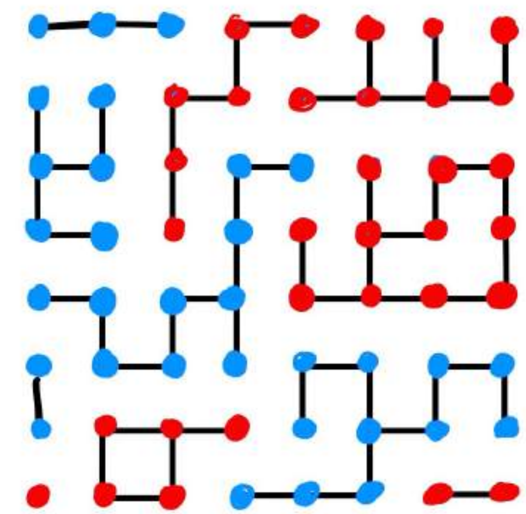
2) The scaling limit is conformal invariant.

$d = 2$ $\quad \Delta' = \dfrac{5}{48}$ $\quad SLE_6 \longleftarrow$ CARDY, DUPLANTIER, SALEUR
SCHRAMM, LAWLER, WERNER, SMIRNOV

$d = 3, 4, 5$: "interacting CFT" $\quad \Delta'(3) \approx 0.477$.

$d \geq 6$ $\quad \Delta' = \dfrac{d-2}{2}$

Critical percolation $\longrightarrow$ conf. inv. measure on $\mathcal{D}'(S^d)$



Divide-and-color HÄGGSTRÖM, CAMIA

① sample edges according to critical percolation

② "color" vertex clusters $\sigma = \pm 1$ with eq. prob., independently at random

$$\Rightarrow \quad \mathbb{E}\left(\sigma(x)\, \sigma(y) \right) = \mathbb{P}(x \sim y) \sim \frac{B(d)}{|x-y|^{2\Delta'(d)}}$$

$$\phi = \lim_{a \to 0} \frac{1}{\sqrt{B(d)}}\, a^{d-\Delta'(d)} \sum_{x \in \mathbb{Z}^d} \sigma(x)\, \delta_{ax}$$

expected to be a conformally-invariant random element of $\mathcal{D}'(S^d)$

$\mathrm{CFT}_d \approx$ "study of conformally-invariant measures on $\mathcal{D}'(S^d)$"

What does $\approx$ mean?

$\approx \neq \subseteq$: fermions, conformal gauge theories

$\approx \neq \supseteq$: $\exists$ non-CFT-like conf. inv. measures on $\mathcal{D}'(S^d)$

to address the 2^{nd} point, impose additional constraints

Definition (Schwinger functions):

Let μ be a Δ-conformal-inv. measure on $\mathcal{D}'(S^d)$.

The n^{th} Schwinger function is the distribution on $\mathbb{R}^{nd}$

$$S_n(f_1, \dots, f_n) := \int_{\mathcal{D}'(S^d)} \phi(f_1) \dots \phi(f_n) \, d\mu(\phi)$$

Proposition: S_n is Δ-conformal invariant.

Corrolary: $S_1(f_1) = 0$

$$S_2(f_1, f_2) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{f_1(x_1) f_2(x_2)}{|x_1 - x_2|^{2\Delta}} \, dx_1 \, dx_2 .$$

Reflection-positivity : $\quad \mathbb{E}\left(\frac{\cdot\phi \quad \cdot\phi \quad \cdot\phi}{\cdot\phi \quad \cdot\phi \quad \cdot\phi} \right) \geq 0 \qquad \uparrow x^d$

<u>Definition:</u> A measure μ on $S'(\mathbb{R}^d)$ is reflection-positive if for all sequences $f_0, \dots, f_n$; $f_j \in S(\mathbb{R}^{dj})$, $\operatorname{supp}(f_j) \subseteq \mathbb{R}_-^{dj}$, we have $\sum_{j,k=0}^{n} S_{j+k}(f_j \otimes \overline{\Theta f_k}) \geq 0$. $\qquad \Theta : x_d \mapsto -x_d$

Typically follows from
1) positivity of μ
2) transl.-inv. Hamiltonian
3) locality (Markov property)

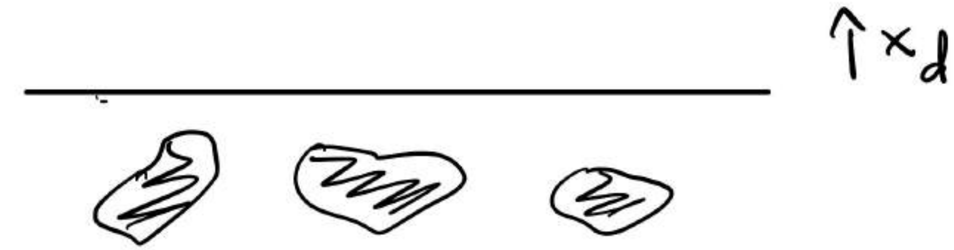
$\left\{ \begin{array}{ll} \text{Ising} & ✓ \\ \text{percolation} & ✗ \end{array} \right.$

OSTERWALDER + SCHRADER: Reflection-positive theories have a natural Hilbert space

$$\mathcal{H}_0 = \langle (f_0, \dots, f_n) \;:\; f_j \in \mathcal{S}(\mathbb{R}_+^{dj}),\; n \in \mathbb{N} \rangle$$

seminorm $$\|(f_0, \dots, f_n)\|^2 = \sum_{j,k=1}^{n} S_{j+k}(f_j\, \overline{\Theta f_k}) \geq 0$$

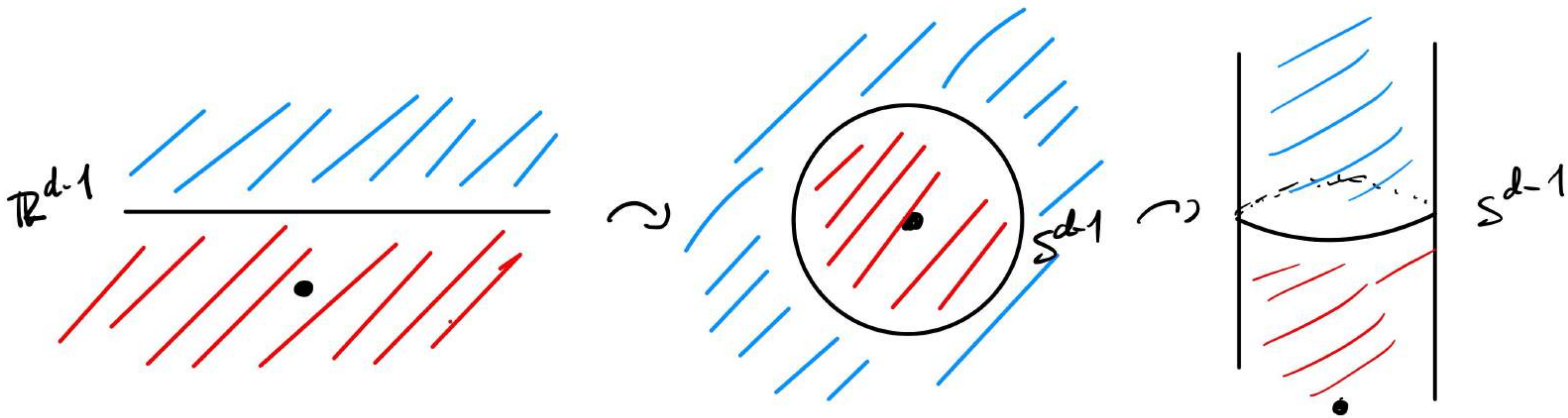
$$\mathcal{H} = \text{completion}\left(\mathcal{H}_0 / \text{null vectors}\right)$$



LÜSCHER + MACK: If we start from a conformally-inv. measure,

$\mathcal{H}$ is a positive-energy, unitary representation of $SO(2,d)$.

why $so(2,d)$? $\mathrm{conf}(\underbrace{\mathbb{R}}_{-} \times \underbrace{S^{d-1}}_{+\cdots+}) = so(2,d)$



- $D :=$ Hermitian generator of $so(2)$ in $so(2) \oplus so(d) \subset so(2,d)$.
- Positive energy $\Leftrightarrow$ $D \geq 0$.

States

irreducible, pos. en., unitary rep. of $so(2,d)$

- $$\mathcal{H} \cong \underbrace{\mathbb{C}}_{\text{vacuum } \Omega} \oplus \bigoplus_{j=1}^{\infty} R_{\Delta_j, \rho_j}$$

spin ρ = irrep of $so(d)$

scaling dimension $\Delta \in \mathbb{R}_{\geq 0}$ (D eigenvalue)

under $so(2,d) \to so(2) \oplus so(d)$

$$R_{\Delta,\rho} \cong \underbrace{(\Delta, \rho)}_{\text{primary}} \oplus \underbrace{(\Delta+1, \rho') \oplus (\Delta+2, \rho'') \oplus \dots}_{\text{desendants}}$$

Theorem:

$$\mathcal{H}(\text{Gaussian free field}) \cong \bigoplus_{n=0}^{\infty} \mathrm{Sym}^n\left(R_{\frac{d-2}{2},0}\right)$$

Idea of proof: $\mathrm{Sym}^n\left(R_{\frac{d-2}{2},0}\right) \subset \mathcal{H}$ generated by "normal-ordered product"

$n=0$ $\quad f_0 \to \mathbb{C}$ = vacuum

$n=1$ $\quad \phi(f_1) \to R_{\frac{d-2}{2},0}$

$n=2$ $\quad \phi(f_1)\phi(f_2) - \mathbb{E}(\phi(f_1)\phi(f_2)) \to \mathrm{Sym}^2\left(R_{\frac{d-2}{2},0}\right)$

Operators

$\mathcal{H}$ comes equipped with a canonical infinite set of Wightman fields $\Phi_j : \mathcal{S}(\mathbb{R}^{1,d-1}) \to \mathcal{L}(\mathcal{D}, \mathcal{D})$ with a common dense domain $\mathcal{D} \subset \mathcal{H}$, such that

1) $\Phi_j(f)^* = \Phi_j(f)$

2) $\Phi_j(f)\Omega \in \mathcal{R}_{\Delta_j, \ell_j} \subset \mathcal{H}$

3) $[\Phi_j(f_1), \Phi_k(f_2)] = 0$ if $\mathrm{supp}(f_1), \mathrm{supp}(f_2)$ spacelike.

4) $\forall g \in \widetilde{SO}(2,d)$: $U(g)\Phi_j(f)U(g)^* = \Phi_j(g \cdot f)$

Matrix elements of Φ_j $\Leftrightarrow$ OPE coefficients $\Leftrightarrow$ three-point functions

<u>Proposition:</u> Let $v_i \in R_{\Delta_i, \ell_i} \subset \mathcal{H}$, $v_k \in R_{\Delta_k, \ell_k} \subset \mathcal{H}$.

Then $(v_i, \Phi_j(f)\, v_k)_{\mathcal{H}}$ is uniquely determined by conformal symmetry (as a function of v_i, v_k, f) up to finitely many constants

$c_{ijk}^{\alpha} \in \mathbb{R}$ $\quad \alpha = 1, \dots, N(\ell_i, \ell_j, \ell_k)$.

<u>Idea of proof:</u> The conformal group acts transitively on triples of points.

The conformal bootstrap / crossing equation

POLYAKOV, MIGDAL
FERRARA, GATO, GRILLO

- Let $v_i \in R_{\Delta_i, \rho_i} \subset \mathcal{H}$, $v_m \in R_{\Delta_m, \rho_m} \subset \mathcal{H}$

- $\mathrm{supp}(f_1)$, $\mathrm{supp}(f_2)$ spacelike $\Rightarrow (v_i, [\Phi_j(f_1), \Phi_k(f_2)] v_m) = 0$

$$\Leftrightarrow \quad (v_i, \Phi_j(f_1)\Phi_k(f_2) v_m) = (v_i, \Phi_k(f_2)\Phi_j(f_1) v_m)$$

- Insert spectral resolution of identity

$$\boxed{\sum_{n=0}^{\infty} \sum_{\alpha,\beta} C^{\alpha}_{ijn} C^{\beta}_{nkm} G^{n,\alpha,\beta}_{ijkm} = \sum_{n=0}^{\infty} \sum_{\alpha,\beta} C^{\alpha}_{ikn} C^{\beta}_{njm} G^{n,\alpha,\beta}_{ikjm}}$$

- Conformal blocks $G^{n,\alpha,\beta}_{ijkm}(v_i, v_m; f_1, f_2)$ fixed by conf. sym.

An ∞ set of eqn^s for ∞ many unkowns (Δ_i, ℓ_i), c_{ijk}^{α}.

<u>Comments</u>:

- Measure on $\mathcal{D}'(S^d)$ disappeared from the description.
- Same eqn^s expected to hold also with fermions & gauge fields.
- For $d=3$ Ising, they imply tight bounds on Δ_i, c_{ijk}.

  RYCHKOV ET AL
- Followed OSTERWALDER + SCHRADER $\longrightarrow$ WIGHTMAN $\longrightarrow$ BOOTSTRAP

  Can follow BOOTSTRAP $\longrightarrow$ WIGHTMAN KRAVCHUK, QIAO, RYCHKOV '20, '21

Reflection-positive CFT_d s $\simeq$ solutions of the crossing equations.

Conformal bootstrap in $d \geq 3$ without reflection positivity?

→ percolation, self-avoiding walks

Reflection positivity $\Rightarrow$ norm preserved by $SO(2,d)$

$\Rightarrow$ spectral resolution with positive coefficients

<u>Idea</u>: Replace reflection positivity by probabilistic positivity.

- Replace $\mathcal{H}$ by $L^2(D'(S^d), \mu_{\text{critical}})$.
- Norm on $L^2(D'(S^d))$ preserved by $\text{Conf}^+(S^d) \cong SO(1,d+1)$.
- $SO(2,d)$ unitarity $\rightarrow$ $SO(1,d+1)$ unitarity
- spectrum : $L^2(D'(S^d)) = \mathbb{C} \oplus C_{\Delta(d)} \oplus$ continuous spectrum.

  complementary series
- Replace $\Phi_j(f)$ by $\phi(f)$.
- $\phi(f_1)\phi(f_2) = \mathbb{E}(\phi(f_1)\phi(f_2)) + c_{\phi\phi\phi}\,\phi(f_1 * f_2) + \text{cts spectrum}$

  $c_{\phi\phi\phi} \in \mathbb{R}$ (triple connectivity)

- $\mathbb{E}(\phi(f_1)\phi(f_2)\phi(f_3)\phi(f_4)) = \mathbb{E}(\phi(f_1)\phi(f_2))\mathbb{E}(\phi(f_3)\phi(f_4)) +$

$+ \ c^2_{\phi\phi\phi} \ \mathcal{G}^{\phi\phi\phi\phi}_{\phi}(f_1, f_2, f_3, f_4) +$ continuous spectrum

- permutation symmetry $\Rightarrow$ constraints on Δ, $c^2_{\phi\phi\phi}$

$\overset{?}{\Rightarrow}$ bounds on triple connectivity in Percolation (d)

WIP with KRAVCHUK, PAL,
RADCLIFFE, ZHAO

KRAVCHUK
DM
PAL '21

conformal bootstrap $\Leftrightarrow$ spectral theory of automorphic forms

- Hyperbolic $(d+1)$-manifold $M = {}_{\Gamma}\backslash \mathbb{H}^{d+1}$, $\Gamma < SO(1,d+1)$.
- ${}_{\Gamma}\backslash^{SO(1,d+1)}$ toy model of $(\mathcal{D}'(S^d), \mu_{\text{critical}})$
- $L^2\left({}_{\Gamma}\backslash^{SO(1,d+1)}\right) \cong \bigoplus$ (automorphic forms)
- $c_{ijk} \Leftrightarrow$ L-functions (for $d=1$).
- associative multiplication in $C^\infty\left({}_{\Gamma}\backslash^{SO(1,d+1)}\right) \Leftrightarrow$ bootstrap

conformal bootstrap with probabilistic positivity

$\Rightarrow$ tight bounds on Laplace spectra BONIFACIO, KRAVCHUK, DM, PAL '21

$\Rightarrow$ Weyl bound on triple product L-functions ADVE, BONIFACIO, KRAVCHUK, DM, PAL, RADCLIFFE, ROGELBERG '25

<u>Theorem (Adve 2025):</u> (converse theorem for the bootstrap)

$$\left\{ \begin{array}{l} \text{solutions of the } SO(1,2) \\ \text{probabilistic conf. boot.} \\ \text{w/ discrete spectra} \end{array} \right\} = \left\{ \Gamma \backslash SO(1,2) \;,\; \Gamma \text{ discrete cocompact} \right\}.$$

Goals / dreams for the future

① Make the bootstrap definition of CFT_d completely precise.

② Identify the analogue of $\Gamma \backslash SO(1,d+1)$, providing models.

③ Prove the converse theorem.