Invariant Gibbs measures and propagation of randomness for nonlinear dispersive PDE

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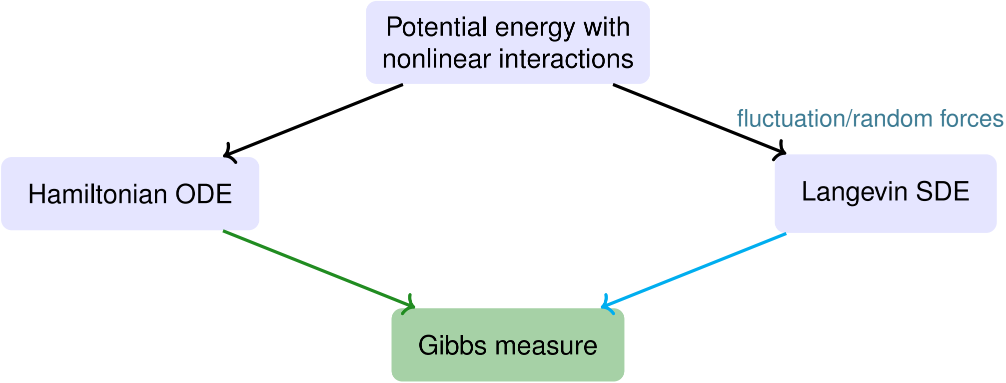
Will mention some aspects of:

§IV. Invariance of the 3D Gibbs measure under the cubic NLW (the hyperbolic Φ^4_3);

joint work with Bjoern Bringmann, Yu Deng and Haitian Yue

Hamiltonian ODE and Langevin SDE

- Physical system



Gibbs Measure on Tori

Consider $\phi : \mathbb{T}^d_x \to \mathbb{R}$ or $\mathbb{C}$, $\quad p \geq 3$, odd and an 'energy' functional

$$H(\phi) = \int_{\mathbb{T}^d_x} \left(\frac{|\phi|^2}{2} + \frac{|\nabla \phi|^2}{2} + \frac{|\phi|^{p+1}}{p+1} \right) dx.$$

- Formally, we can associate it to a Gibbs measure:

$$d\mu(\phi) \sim Z_\beta^{-1} \exp\left(-\beta H(\phi) \right) \prod d\phi(x)$$

as well as 3 different dynamical flows $\quad \longleftrightarrow \Phi^{p+1}_d$ models:

(a) A nonlinear Schrödinger equation. ($\leftrightarrow$ complex-valued Hamiltonian)

(b) A nonlinear wave equation. ($\leftrightarrow$ real-valued Hamiltonian)

(c) A nonlinear stochastic heat equation. ($\leftrightarrow$ Langevin)

Literature of Φ_d^{p+1} models on $\mathbb{T}^d$

Dimension	Measure	Heat	Wave	Schrödinger	
$d = 1$	$(p \geq 3)$	[Iwa87]	[Fri85]	[Bou94]	
$d = 2$	[Nel66] $(p \geq 3)$	[DPD03]	[Bou99]	[Bou96] $(p = 3)$	**[DNY19]**
$d = 3$	[GJ73] $(p = 3)$	[Hai13] [HM18, MW20]	**[BDNY22]**	Open	
$d = 4$	[ADC19]				
$d \geq 5$	[Aiz81, Fro82]				

Timeline: 2022

1960 2026

The NLS on $\mathbb{T}^d$

$$\begin{cases} iu_t + \Delta u = |u|^{p-1}u \\ u(x,0) = u(0), \qquad x \in \mathbb{T}^d \end{cases}$$

Here $p \geq 3$ is odd, eq. is defocusing, conserves mass (L^2-norm) and Hamiltonian:

$$H(u)(t) = \int_{\mathbb{T}^d_x} \left(\frac{|u|^2}{2} + \frac{|\nabla_x u|^2}{2} + \frac{|u|^{p+1}}{p+1} \right) dx.$$

Deterministic scaling: $s_{cr} := \frac{d}{2} - \frac{2}{p-1}$ (heuristics from high-high-to-high)

(Big) Theorem For $s_{cr} \geq 0$, NLS on $\mathbb{T}^d$ is locally well-posed for data in H^s when $s > s_{cr}$. Ill-posedness may occur for $s < s_{cr}$.

(Bourgain '93, Bourgain-Demeter '15; Christ-Colliander-Tao '03 . . .)

In BEC, binary collisions between the bosons, yields the cubic NLS on $\mathbb{T}^3$; while ternary collisions give a quintic NLS in $d = 1, 2, 3$ (L. Erdös–B. Schlein–H.T. Yau; T. Chen–N. Pavlović; Y. Hong–K. Taliaferro–Z. Xie; X. Chen-J. Holmer.)

The NLS on $\mathbb{T}^d$: Random data theory

Canonical random data:

$$u^\omega(0) = f(\omega) = \sum_{k\in\mathbb{Z}^d} \frac{g_k(\omega)}{\langle k\rangle^{\alpha}} e^{ik\cdot x}, \qquad \boxed{\alpha := s + \frac{d}{2}}$$

where $\{g_k\}$ are i.i.d . r.v. complex Gaussian $\mathbb{E}g_k = 0$, $\mathbb{E}|g_k|^2 = 1$, or uniformly distributed on the unit circle of $\mathbb{C}$.

The law of $f(\omega)$ is formally given by a Gaussian measure supported on $H^{s-}(\mathbb{T}^d)$, $\boxed{s = \alpha - \frac{d}{2}}$

$\boxed{\alpha = 1}$ special:

- Corresponds to statistical ensemble of Gibbs measures.
- In 2D and 3D such Gibbs measures are supported on distributions: $H^{0-}(\mathbb{T}^2)$ and $H^{-\frac{1}{2}-}(\mathbb{T}^3)$ respectively.

Study of propagation of randomness

Start with random initial data distributed according to some canonical law (e.g. Gaussian, independent Fourier coefficients). *How does this random structure get transported when one moves along the flow of a nonlinear dispersive equation*?

Some natural questions:

- 1. What is the optimal regime where the solution exists and is unique almost surely, at least locally in time?
- 2. Can one describe the solution in terms of the random structure of the initial data -for short times?
- 3. If there are (formally invariant) Gibbs measures, can we justify their invariance?

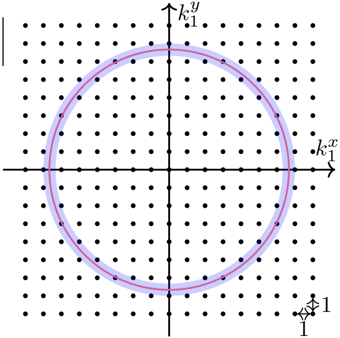
Probabilistic scaling: a guiding principle

$$u^\omega(0) = f(\omega) = N^{-\alpha} \sum_{|k|\sim N} g_k(\omega) e^{ik\cdot x},$$

then f and the linear solution $u^{(1)} := S(t)f(\omega)$ have a.s. unit size in H^s. Let $u^{(2)}$ satisfy $iu_t^{(2)} + \Delta u^{(2)} = |u^{(1)}|^{p-1} \cdot u^{(1)}$. If NLS is a.s. locally well posed, the iteration $u^{(2)}$ should also be bounded in H^s for fixed time t. Fix $|t| \sim 1$ and $|k| \sim N$,

$$\left|\widehat{u^{(2)}}(t,k)\right| \sim N^{-p\alpha} \left| \sum_{k_1,\dots,k_p \in \mathbb{Z}^d,\, |k_j|\sim N} \mathbf{h}^{\mathrm{b}}_{kk_1\cdots k_p} g_{k_1}(\omega)\overline{g_{k_2}}(\omega)\cdots g_{k_p}(\omega) \right|$$

$$\lesssim N^{-p\alpha} \|\mathbf{h}^{\mathrm{b}}_{kk_1\cdots k_p}\|_{\ell^2_{k_1 k_2 \cdots k_p}} \lesssim N^{-p\alpha} \left(N^{d(p-2)+d-2+}\right)^{\mathbf{1/2}}$$



base tensor: $\quad \mathbf{h}^{\mathrm{b}}_{kk_1\cdots k_p} = \mathbf{1}_{\left\{ \substack{k_1 - k_2 + k_3 - \cdots + k_p = k \\ |k_1|^2 - |k_2|^2 + |k_3|^2 - \cdots + |k_p|^2 - |k|^2 \approx \Omega} \right\}}$

so $u^{(2)}$ is bounded in H^s norm if and only if $\boxed{s > -\dfrac{1}{p-1} =: s_{pr}}$

Parabolic and Hyperbolic Comparisons

- Stochastic heat equation:

$$(\partial_t - \Delta)u + u^p = \xi,$$

where ξ is some spacetime Gaussian noise on $\mathbb{R} \times \mathbb{T}^d$; $\quad s_{pr}^H = -\frac{2}{p-1}$

- Nonlinear wave equation:

$$(\partial_t^2 - \Delta)u + u^p = 0,$$

with initial data being a Gaussian random field on $\mathbb{T}^d$; $\quad s_{pr}^W = -\frac{3}{2(p-1)}$.

- The probabilistic scaling heuristics provide a **guiding principle** to the a.s. well-posedness problem of the corresponding dynamics, and should not be understood as an actual threshold.

- In most cases this is indeed true (e.g. NLS $p \geq 3$) but in some cases such as low dimensions and/or low degree nonlinearity, discrepancies may occur.

- Global geometry and specific dispersive relation matter.

Truncated (Wick ordered) cubic NLS on $\mathbb{T}^2$ and probabilistic LWP

- Given a cutoff χ, the truncated Wick ordered cubic NLS on $\mathbb{T}^2$:

$$\begin{cases} (i\partial_t + \Delta)u_N = :|u_N|^2 u_N: \quad (= |u_N|^2 u_N - 2\sigma_N u_N) \\ u_N(0,x) = \mathbb{P}_{\leq N} f^\omega(x) = \sum_{k \in \mathbb{Z}^2} \chi\left(\frac{k}{N}\right) \frac{g_k(\omega)}{\langle k \rangle} e^{ik\cdot x} \end{cases} \qquad \sigma_N = \mathbb{E}\int_{\mathbb{T}^2} |\mathbb{P}_{\leq N} f^\omega|^2 dx \sim \log N.$$

  Suppose the random data f^ω is in $H^{s-}(\mathbb{T}^d)$.

- Probabilistic LWP means: for $0 < \tau \ll 1$, there exists a probability set Z with[1] $\mathbb{P}(Z) \leq C_\theta e^{-\tau^{-\theta}}$ such that when $\omega \notin Z$, the sequence $\{u_N\}$ converges as $N \to \infty$ to a unique limit $u \in C_t^0([-\tau,\tau]; H^{s-}(\mathbb{T}^d))$. Moreover the nonlinearity $:|u|^2 u:$ is well defined in the limit in the sense of distributions and u solves the equation in the distributional sense.

[1] $\theta = \theta(d,p,s)$ is a small constant, independent of τ; C_θ is a constant determined by θ.

Bourgain's method ('96). Probabilistic LWP[2]

- He considered the invariance of the Gibbs measure for the (Wick ordered) cubic NLS equation on $\mathbb{T}^2$. Now $s_{pr} = -\frac{1}{2} < s_G- = 0- < 0 = s_{cr}$.

- Bourgain's main idea is to make a linear-nonlinear decomposition Ansatz , $\boxed{u = e^{it\Delta} f(\omega) + v}$ where the linear part is rough but random, and show the nonlinear part v is smoother (positive regularity).

- Solve the difference equation via a Banach fixed point argument:

$$iv_t + \Delta v = \mathcal{C}(\underbrace{e^{it\Delta} f(\omega)}_{\text{R:=rough-random}} + \underbrace{v}_{\text{smoother-deterministic=:D}})$$

<u>Tools</u>:

  - multilinear large deviation estimates
  - integer lattice counting estimates $\leftrightarrow$ analytic number theory
  - TT^* arguments $\leftrightarrow$ random matrix estimates (way to exploit randomness in absence of gain of regularity).

[2] Bourgain's globalization argument then kicks in. Not trivial or immediate but well understood now.

The DRR case

To close the fixed point argument, need to control all nonlinear interactions RRR, RDD, RRD, DDD and all frequencies' positions. Ignoring the time variable, in one case Bourgain is led to estimate cubic interactions of high frequency deterministic and low frequency random terms:

$$\Big\| \sum_{\substack{k=k_1-k_2+k_3 \\ |k|^2=|k_1|^2-|k_2|^2+|k_3|^2 \\ k_2\neq k_1,k_3,\, k_j\sim N_j}} \underbrace{\widehat{v}(k_1)}_{\text{high}} \underbrace{\frac{\overline{g_{k_2}(\omega)}}{\langle k_2\rangle} \frac{g_{k_3}(\omega)}{\langle k_3\rangle}}_{\text{low}} \Big\|_{\ell^2_k}$$

- We cannot apply LDE directly to compute the ℓ^2_k norm above.
- Hölder doesn't work: decay from the random terms is too weak to offset the counting estimates
- Need to find correct way to exploit randomness/independence in the absence of smoothing:
  - Bourgain's idea is to use a TT^* argument $\leftrightarrow$ good random matrix estimate. He considers $\mathcal{G}=\mathcal{G}_\omega=(\sigma_{k,k_1})$ acting on the vector $\widehat{v}(k_1)$ defined as :

$$\sigma_{k,k_1} := \frac{1}{N_2N_3} \sum_{k_2,k_3 \text{ as above}} \overline{g_{k_2}(\omega)}g_{k_3}(\omega),$$

and proves that $\|\mathcal{G}\mathcal{G}^*\|_{OP}^{1/2}$ has the right decay to close.

Bourgain's globalization argument

Bourgain's general strategy for proving global well-posedness and invariance, consists of the following four steps:

I. Start with a.s LWP for the truncated evolution equation, and consider the corresponding truncated Hamiltonian and associated Gibbs measure. Due to ODE-methods, the truncated Gibbs measure is invariant[3] under the truncated evolution equation.

II. Hence using (I) one obtains global bounds for the truncated dynamics. In this step, it is essential that the global bounds are uniform in the truncation parameter. The global bounds must control not just the solution itself, but also the individual components of the Ansatz.

III. Next, one proves the global well-posedness of the full evolution equation using the global bounds from (II), commutator and stability arguments (under small smooth perturbations of the data).

IV. Finally, one proves the invariance of the Gibbs measure under the full evolution equation using the global well-posedness from (III) and the invariance for the truncated evolution equation.

(II) is hardest when implementing Bourgain's globalization argument.

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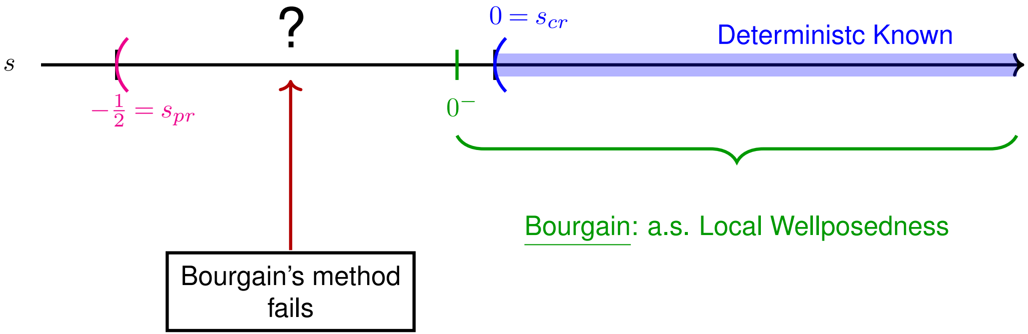
[3] It acts as a conservation law on the statistical ensemble.

Cubic NLS on $\mathbb{T}^2$



- Why: If we start with random data f a bit rougher than Gibbs and proceed with linear-nonlinear ansatz:

$$u = e^{it\Delta} f(\omega) + v.$$

Then remainder v is not as regular as before (stays below L^2).

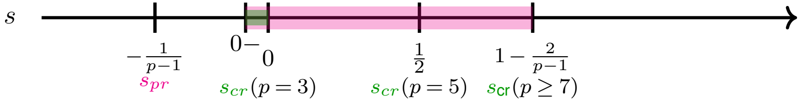
Similar problem for p-NLS on $\mathbb{T}^2$ for any $p \geq 5$

- If we consider -say- $p = 5$, and recall $s_{cr} = \frac{1}{2}$ and following Bourgain write

$$u = e^{it\Delta} f(\omega) + v$$

then $e^{it\Delta} f(\omega) \in H^{0-}$ but v can only be put in $H^{\frac{1}{2}-}$ which still (det.) supercritical: one cannot close the estimate by itself.

- In fact on $\mathbb{T}^2$, any $p \geq 3$ odd, to prove the invariance of Gibbs measure and existence of global strong solutions in its statistical ensemble we must overcome:



- So one cannot close the estimates of the fixed point argument as in Bourgain by relying solely on the (poor) regularity of v.

- We need to understand the intrinsic random structure of v. First question is where does this poorer regularity of v comes from?

- Just as it was the case in the study of singular stochastic heat equation (by Hairer's regularity structures or Gubinelli et al.'s paracontrolled calculus) the culprits are HL-H wave interactions. In our case, and in its simplest form:

$$(u^{\omega}_{\text{lin}})_{\text{high}}\, u_{\text{low}}\, u_{\text{low}} \qquad \longleftarrow \text{need to remove (but, all of them!)}$$

If we attempt to proceed as in the theory of paracontrolled calculus, we'd try to identify a term X from v that is paracontrolled by u_{lin}; that is (up to some smooth remainder) :

$$X := \mathcal{I}\pi_{>}(u_{\mathrm{lin}}, :|u|^{p-1}:) \qquad \mathcal{I} = (i\partial_t - \Delta)^{-1}$$

and hope that X behaves like u_{lin} and, that $Y := v - X$ is smoother.

- But for this, we need some control on the lower frequency part of $:|u|^{p-1}:$ which itself contain $|X|^{p-1}$ whose regularity is $H^{\frac{1}{2}-}$, still supercritical $\rightarrow$ no way of controlling it assuming only this (its regularity).
  - For heat -and to some extent wave- X has higher regularity due to smoothing; so the low frequency part above is a nice function and can be place directly into a good function space.
  - Instead one needs to **zoom in/unveil and invoke the random structure** of X.
  - But how can this be done and done in a unified manner for all $p \geq 5$?.

The Random Averaging Operators (RAO)

The goal is to capture the implicit randomness structure of $P_N X$ in some norm or quantity that propagates[4]; i.e. that allows for an induction in frequencies from low frequency $L \ll N$ to frequency N.

- Clearly the $H^{\frac{1}{2}-}$ norm of $P_N X$ will not be enough to do the job, since this is a supercritical norm (for any $p \geq 5$) .
- It turns out that, to find the right quantity, we need to **shift the point of view** from the **term**

$$P_N X = \sum_{L \ll N} \underbrace{\mathcal{I}(P_N u_{\text{lin}} \cdot :|P_L u|^{p-1}:)}$$

to the **operator**

$$\mathcal{P}_{NL} : w \mapsto \mathcal{I}(P_N w \cdot :|P_L u|^{p-1}:).$$

[4]Propagates means that if you can prove the bounds for the low frequency part then you can prove the bound for the higher frequency part.

The full RAO ansatz

Now we can write down the full ansatz of the solution to the NLS with $p \geq 5$ with data in the statistical ensemble of the Gibbs measure:

$$u = u_{\text{lin}} + \sum_N \sum_{L \ll N} \mathcal{P}_{NL}(u_{\text{lin}}) + w,$$

where $\mathcal{P}_{NL}$ are the random averaging operators, whose coefficients are shown (after some analysis) to be independent with $P_N u^\omega_{\text{lin}}$.

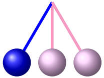
(a) The L^2-operator norm and Hilbert-Schmidt norm precisely capture the random structure of $\mathcal{P}_{NL}$.

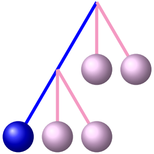
(b) We induct on frequency to show that: 1) $\mathcal{P}_{NL}$ satisfy two suitable (a priori) bounds and, 2) that the remainder w is bounded in the subcritical space[5] H^{1-}.

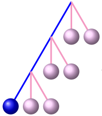
Let's summarize diagrammatically (in the cubic case for simplicity) what we just described. Let $0 < \delta < 1$ small.

[5] more precisely some $X^{1-,b}$ space.

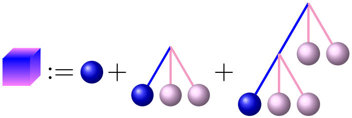
Let $\mathcal{I} =$ Duhamel operator. Denote  $:= e^{it\Delta} f_N(\omega)$ and  $:= u_{N^\delta}$.

 $:= \mathcal{I}\mathcal{C}(e^{it\Delta} f_N(\omega), u_{N^\delta}, u_{N^\delta}),$

 $:= \mathcal{I}\mathcal{C}\big(\mathcal{I}\mathcal{C}(e^{it\Delta} f_N(\omega), u_{N^\delta}, u_{N^\delta}), u_{N^\delta}, u_{N^\delta}\big),$

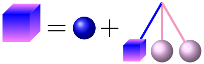
and so on  $, \cdots$.

The sum of these trees forms on an infinite series of trees which is equivalent to a para-linearized equation that can be explicitly solved; gives rise to the method of random averaging operators (RAO). Indeed:

$$\Psi_{N,N^\delta} =$$  $$+ \cdots .$$

is equivalent to a para-linearized equation,

$$\begin{cases} (i\partial_t + \Delta)\Psi_{N,N^\delta} = \mathcal{C}(\Psi_{N,N^\delta}, u_{N^\delta}, u_{N^\delta}); \\ \Psi_{N,N^\delta}(0) = f_N(\omega), \end{cases}$$

$$\iff \Psi_{N,N^\delta} = u_{\text{lin}} + \mathcal{P}_{N,N^\delta}(\Psi_{N,N^\delta}) \iff$$ 

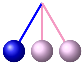
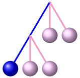
$$\iff \Psi_{N,N^\delta} = (\text{Id} - \mathcal{P}_{N,N^\delta})^{-1}(u_{\text{lin}}),$$

with the k-th Fourier mode of Ψ_{N,N^δ} has the following form:

$$\boxed{\mathcal{F}(\Psi_{N,N^\delta})(k) = \sum_{|k_1|\sim N} h_{kk_1}(t)\frac{g_{k_1}(\omega)}{\langle k_1\rangle^\alpha}}$$

where $h_{kk_1}(t)$ is the $(1,1)$ random tensor mapping $k_1 \to k$ (t can be treated as a parameter) ; independent of $g_{k_1}(\omega)$.

Then the Ansatz for the solution[6] is:

$u = \quad u_{\text{lin}} + \Big(\sum_N$  $+$  $+ \cdots \Big) + \text{remainder}$

$$= \underbrace{u_{\text{lin}}}_{\in H^{0-}} + \underbrace{\mathcal{P}(u_{\text{lin}})}_{\in H^{\frac{1}{2}-}} + \underbrace{\text{remainder}}_{\in H^{1-}}$$

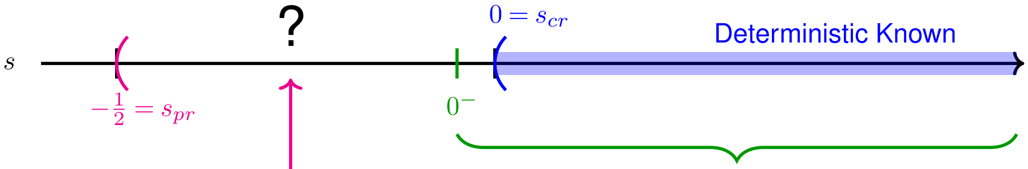
Point: We view the key high-low interactions where the high frequencies come from u_{lin}^{ω} as a random operator $\mathcal{P}$ applied to u_{lin}^{ω}. We expand the solution u in Fourier space, where $u_k(t) := \widehat{u}(t,k)$, as

$$\boxed{u_k(t) = \frac{g_k(\omega)}{\langle k \rangle^{\alpha}} + \sum_{k_1} h_{kk_1}(t) \frac{g_{k_1}(\omega)}{\langle k_1 \rangle^{\alpha}} + (\text{remainder})_k} \qquad \text{(RAO)}$$

where the $(1,1)$ random tensor (matrix) h_{kk_1} contains all the randomness information of the low frequency components of the solution u and satisfies suitable ℓ^2 operator norm estimates as a mapping from $k_1 \to k$.

[6] We also globalize the local-in-time random averaging operator structures: Bourgain's globalization + the stability of the random structures (RAO ansatz) under small H^{1-} perturbations of the data.

More generally how do we prove prob. LWP for NLS in the full subcritical range $s > s_{prob}$?



Bourgain: A.S. Local Wellposedness

RAO $\hookrightarrow$ Random Tensor Theory (Y. Deng, A.N, H. Yue)

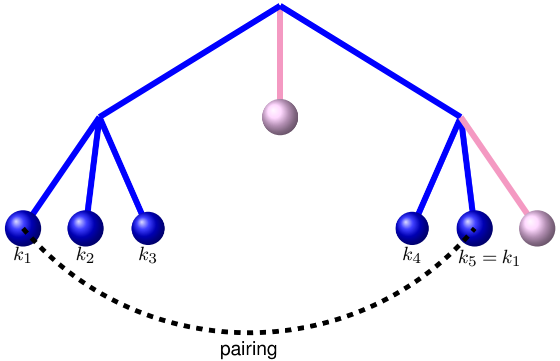
- Unveil the random structure of the remainder v above.
- Detailed information on propagation of randomness.

Works for any $d \geq 2$ and any number of wave interactions $p+1$ (unified theory).

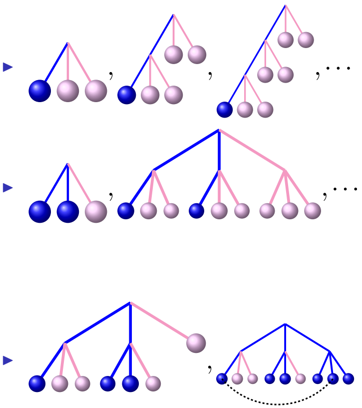
Components of the solutions: all kinds of iteration trees

To find the solution expand nonlinearity using the equation itself; keep expanding until we hit a low frequency input dictated by how close we are to s_{pr} $\rightarrow$ iteration procedure represented by Feynman-type diagrams or tree expansions such as:

- an example iteration tree:



- more trees:



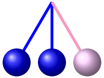
- To find the solution on the full probabilistic subcritical regime $s > s_{pr}$ we need arbitrary long (finite) high-order expansions. This gives rise to what we call random $(q, 1)$ tensors $h = h_{kk_1 \ldots k_q}(t)$ where t is viewed as a parameter, which are highly nonlinear objects arising from the high-order iteration trees.

- The random tensors allow us to get a handle of the exploding complexity that arises from the high-order iteration trees.

- The random tensors carry the (random) information of the low frequency components of the solution and are independent from the Gaussians $g^{\pm}_{k_j}(\omega), j = 1, \ldots, q$. .

Simple example (p=3) of a (2,1) tensor term

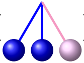
- The $(2,1)$ tensors have 2 terminal leaves which are Gaussian term $e^{it\Delta} f_N(\omega)$; the other terminal leaves are low frequency components u_{N^δ}.
- Denote



$$:= \mathcal{I}\mathcal{C}(e^{it\Delta} f_{N_1}(\omega), e^{it\Delta} f_{N_2}(\omega), u_{N^\delta})$$

where $\mathcal{I}$ is the Duhamel operator, $N = \max(N_1, N_2)$ and $N_1, N_2 > N^\delta$, $\;0 < \delta < 1$ fixed. Then (modulo details about the temporal frequency):

The k-th Fourier mode is

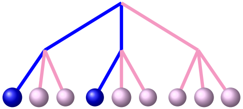
$\mathcal{F}($  $)(k)$

$$\sim \sum_{k_1,k_2} \underbrace{\left(\sum_{|k_3| \leq N^\delta} \mathbf{1}_{\left\{ \substack{k = k_1 - k_2 + k_3 \\ |k|^2 = |k_1|^2 - |k_2|^2 + |k_3|^2} \right\}} \widehat{u}(k_3) \right)}_{h_{kk_1k_2}} \frac{g_{k_1}(\omega)}{\langle k_1 \rangle^\alpha} \frac{\overline{g_{k_2}(\omega)}}{\langle k_2 \rangle^\alpha}$$

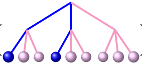
where $|k_1| \sim N_1$, $|k_2| \sim N_2$ and $|k_3| \leq N^\delta$. Note that here $h_{kk_1k_2}$ is a (2,1) random tensor -say- maps $k_1, k_2 \to k$.

Another example:

$$\mathcal{IC}\big(\mathcal{IC}(e^{it\Delta}f_{N_a}, u_{N^\delta}, u_{N^\delta}), \mathcal{IC}(e^{it\Delta}f_{N_b}, u_{N^\delta}, u_{N^\delta}), \mathcal{IC}(u_{N^\delta}, u_{N^\delta}, u_{N^\delta})\big)$$

$:=$ 

where $N = \max(N_a, N_b)$ and $N_a, N_b > N^\delta$.

$\mathcal{F}\Big($  $\Big)(k) = \displaystyle\sum_{\substack{|a|\sim N_a \\ |b|\sim N_b}} h_{kab} \cdot \frac{g_a(\omega)}{\langle a\rangle^\alpha} \frac{\overline{g_b(\omega)}}{\langle b\rangle^\alpha}$

Note that h_{kab} is a $(2,1)$ random tensor which maps $a, b \to k$

Solution ansatz:

- We make the ansatz for the Fourier coefficient: $u_k(t) := \widehat{u}(k,t)$ of the solution:

$$u_k^\omega(t) = \frac{g_k(\omega)}{|k|^\alpha} + \sum_{k_1} h_{kk_1}(t)\, g_{k_1}(\omega) + \sum_{k_1 k_2} h_{kk_1k_2}(t)\, g_{k_1}(\omega)\overline{g_{k_2}}(\omega) +$$

$$\cdots\cdots + \sum_{k_1 k_2 \cdots k_q} h_{kk_1k_2\cdots k_q}(t)\, g_{k_1}(\omega)\overline{g_{k_2}}(\omega)\cdots g_{k_q}(\omega) + (\text{Remainder})_k$$

- The convergence of expansion is completely determined by the properties of these tensors.
- Heuristically the difficulty of covering the full probabilistically subcritical range $s > s_{pr}$ (for fixed p) can be measured by the order of the (finite) expansion needed, which tends to infinity as $s \to s_{pr}$.
- Each iteration of the equation gains regularity $\sim (s - s_{pr})$ (as in probabilistic scaling heuristcs argument).

Random tensors framework

- We develop an algebraic theory: structure of the tensors and how they are built from smaller tensors using certain operations such as multilinear tensor products, contractions with gaussians, etc. giving rise to two algebraic operations: merging and trimming.

- We also develop the analytic theory, which behaves well with our algebraic theory and entails choosing suitable norms for the tensors $h = h_{kk_1 \ldots k_q}$, for which we prove several estimates that provide suitable bounds for the tensor terms and remainder (merging estimates and random matrix estimates.).

- Proof proceeds then by induction relying on the above $+$
  - LDE
  - integer lattice counting estimates,
  - high-order TT^*/random matrix estimates
  - subtle selection algorithm needed to exploit the flexibility we build into the estimates proved in our analytic theory.

RTT Analytic Theory–Norms

Given a set of input variables A, $\quad h_{k_A} : (\mathbb{Z}^d)^A \to \mathbb{C}$ is a function.

For (B, C) a partition of A, we define the operator norm $\|h\|^2_{k_B \to k_C}$ of h viewed as an operator mapping functions of k_B to functions of k_C. For example for a tensor $h = h_{kxyz}$ we define

$$\|h\|^2_{kx\to yz} := \sup\left\{ \sum_{y,z} \left| \sum_{k,x} h_{kxyz} \cdot z_{kx} \right|^2 : \sum_{k,x} |z_{kx}|^2 = 1 \right\}$$

In some instances, we just use the ℓ^2 norm of h in all its variables (Hilbert-Schmidt norm), for example for $h = h_{ab}$, we have

$$\|h\|^2_{ab} = \sum_{a,b} |h_{ab}|^2.$$

A crucial component of the RTT is the choice of norms for the tensors $h = h_{kk_1\ldots k_q}$ that behave well with the algebraic process of merging and trimming.

Analytic theory: merging estimates

Suppose we consider the semiproduct (merged tensor):

$$\mathfrak{h}_{bczw} = \sum_{a,e,f} (h^1)_{abc}(h^2)_{aef}(h^3)_{efzw}$$

Then we have the following multilinear estimates

$$\|\mathfrak{h}\|_{bz\to cw} \leq \|h^1\|_{ab\to c}\|h^2\|_{ef\to a}\|h^3\|_{z\to wef}, \text{ or } \leq \|h^3\|_{efz\to w}\|h^2\|_{a\to ef}\|h^1\|_{b\to ac}, \ \cdots$$

- The formula of $\mathfrak{h}$ does not depend on the order of h^j, but the right hand sides of the inequalities do. So we get a set of inequalities by reordering the tensors, from which we may choose at our disposal.

- Here is where a delicate selection algorithm comes in to optimize the choice one makes in the multilinear merging estimates.

Analytic theory: random matrix estimates

- We also encounter tensors of the form:

$$\mathcal{H}_{kxz} = \sum_{yw} h_{kxyzw} \cdot g_y(\omega)\overline{g_w(\omega)},$$

where the random tensor $h = h_{kxyzw}$ is independent with g_y and g_w, we have with high probability that

$$\|\mathcal{H}\|_{kx\to z} \lesssim N^{\varepsilon} \max\big(\|h\|_{kxyw\to z}, \|h\|_{kxy\to zw}, \|h\|_{kxw\to zy}, \|h\|_{kx\to zyw}\big)$$

where N is the max size of k_{xyzw}, and $\varepsilon > 0$ is arbitrarily small.

  - Proof goes back to Bourgain's '96 paper and relies on **high order** TT^* (random matrix) argument and multilinear estimates above.

Armed with both the algebraic and analytic theory of the RTT we go back and analyze the norms of the tensors appearing in

$$u_k^\omega(t) = \sum_q \sum_{k_1,\cdots,k_q} h_{kk_1\cdots k_q}(t) \prod_{j=1}^{q} \frac{g_{k_j}^{\pm}(\omega)}{\langle k_j \rangle^\alpha} + (\text{remainder})_k \tag{1}$$

These bounds are in fact quite simple. We aim at proving essentially that

$$\|h_{kk_1\ldots k_q}\|_{kk_1\ldots k_r \to k_{r+1}\ldots k_q} \lesssim \prod_{j=1}^{q} N_j^\beta \,(\max_{r+1\le j\le q} N_j)^{-\beta} \tag{2}$$

for any r, where $\langle k_j \rangle \sim N_j$ and $\beta \equiv \alpha -$.

Moreover we prove a Fourier weighted estimate that localizes h as a multilinear Fourier multiplier; i.e. in the support of $h_{kk_1\ldots k_q}$ we have that

$$k \approx \pm k_1 \cdots \pm k_q$$

Hyperbolic Φ_3^4 problem: Invariance of 3D Gibbs cubic NLW

Dimension	Measure	Heat	Wave	Schrödinger	
$d = 1$					
$d = 2$			[Bou99]	$p = 3$ [Bou96]	$p \geq 5$ **[DNY19]**
$d = 3$			**[BDNY22]**	Open	
$d = 4$					
$d \geq 5$					

Invariance of Gibbs measure under 3D cubic NLW dynamics. Challenges:

- It is harder because now $d\mu \perp d\rho \quad \rightarrow$ probabilistic dependent Fourier modes.
- Spatial regularity $-\frac{1}{2}-$. On the hand, the problem is probabilistically subcritical $s_{pr}^{W} = -\frac{3}{4} < -\frac{1}{2}-$

Some current challenges (ongoing work)

- Gibbs measure for 3D cubic NLS.
  - As in 3D NLW: $d\mu \perp d\rho$ (probabilistic dependent Fourier modes), and measure lives in $H^{-\frac{1}{2}-}$.
  - But now $s_{pr} = -\frac{1}{2} \to$ probabilistically critical problem. Log divergences, generally no gain of regularity with each iterations.

  Joint with: B. Bringmann, Y. Deng, and H. Yue

- Out-of-equilibrium long time dynamics.
  - Is there any part of the random description of the solution (RTT) that propagates for longer times (smooth data)? Is it possible to extend for longer times the random structure lying in the high frequencies components?

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- Random Tensor Theory for NLW $\to$ renormalization theory.

  Joint with: Y. Deng, R. Liang, H. Yue, ...

Many thanks for your attention!!